The Black-Scholes Model

We are now ready to transition to the continuous world. All we need to do is to formulate a stochastic process at discrete time steps δt such that we will be able to take a meaningful limit $\delta t \to 0$. We will see that this, ultimately, will be made possible via the Central Limit Theorem and the transition of the binomial distribution to the normal distribution. However, we only can establish convergence, if we have the appropriate scaling of the 'up'- and 'down'-movements of the stock with δt. It will turn out that the following scaling is appropriate: We will derive the famous Black-Scholes formula in

$$
\begin{tikzcd}
 & s_u = s_{now} \cdot \mathrm{e}^{\mu \delta t + \sigma \sqrt{\delta t}} \\
s_{now} \arrow[ur] \arrow[dr] & \\
 & s_d = s_{now} \cdot \mathrm{e}^{\mu \delta t - \sigma \sqrt{\delta t}}
\end{tikzcd}
$$

four steps:

1. Characterize a stock process on a tree with N steps, such that we can take later the limit $N \to \infty$.
2. Compute the risk-neutral probability q using a Taylor expansion.
3. Use the Central Limit Theorem in order to write the option price as an expectation with respect to a Gaussian probability density.
4. Rewrite the integral in terms of the cumulative normal distribution function.

Characterization of the stock process: Consider such a process with many time steps, such that $N \cdot \delta t = T$, where T is the time of maturity of the option. In this tree, with N time-ticks, each path corresponds to a possible realization of the stock process. Let's define a random variable X_N that counts the number of 'up'-jumps. Then, of course, we will have $N - X_N$ 'down'-jumps. The value at the end node of the tree for a path with X_N 'up'-jumps and $N - X_N$ 'down'-jumps is given by

$$
\begin{aligned}
S_T &= S_0 \exp\left(N \delta t \cdot \mu + \sigma \sqrt{\delta t} (X_N - (N - X_N))\right) \\
&= S_0 \exp\left(\mu T + \sigma \sqrt{T} \frac{2 X_N - N}{\sqrt{N}}\right) \qquad (1)
\end{aligned}
$$

Note that, under the above assumptions, the value of the stock at the end node only depends on the number of 'up' and 'down' movements, not on the particular path that the stock took. Therefore the random variable X_N is sufficient to characterize all possible scenarios of the stock path.

Taylor expansion of the risk-neutral probability: As usual, we obtain the risk-neutral probability using the basic formula

$$q = \frac{s_{now}\mathrm{e}^{r\delta t} - s_d}{s_u - s_d} = \frac{\mathrm{e}^{r\delta t} - \mathrm{e}^{\mu\delta t - \sigma\sqrt{\delta t}}}{\mathrm{e}^{\mu\delta t + \sigma\sqrt{\delta t}} - \mathrm{e}^{\mu\delta t - \sigma\sqrt{\delta t}}}$$

Now we make use of the fact that δt is small. The resulting approximation for q is slightly tricky to compute as the terms above involve $\sqrt{\delta t}$, but we already did this earlier and the final result of the computation is

$$q \approx \frac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \frac{\sigma^2}{2} - r}{\sigma}\right)\right). \tag{2}$$

Application of the Central Limit Theorem: Let us write

$$S_T = S_0\,\mathrm{e}^Y, \qquad Y = \mu T + \sigma\sqrt{T}\left(\frac{2X_N - N}{\sqrt{N}}\right) \tag{3}$$

Clearly, X_N has a binomial distribution and, therefore, under the measure $\mathbb{Q}$ we have

$$\mathbb{E}_{\mathbb{Q}}(X_N) = \mathbb{E}(X_N) = Nq, \qquad \mathrm{Var}(X_N) = Nq(1-q) \tag{4}$$

We compute now the expectation and the variance of Y. First, we see that

$$\begin{aligned}
\mathbb{E}\left(\frac{2X_N - N}{\sqrt{N}}\right) &= \frac{2Nq - N}{\sqrt{N}} \\
&= -\frac{\sqrt{\delta t}}{\sqrt{N}}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right)\cdot N \\
&= -\sqrt{T}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right)
\end{aligned}$$

Therefore, for the expectation of Y (under the measure $\mathbb{Q}$), we find

$$\begin{aligned}
\mathbb{E}(Y) &= \mu T - \sigma\sqrt{T}\sqrt{T}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right) \\
&= (r - \sigma^2/2)T
\end{aligned}$$

The variance, on the other hand, is slightly easier to compute in this approximation as we need to take into account only the leading order term ($q \approx 1/2$). Thus we find

$$\mathrm{Var}\left(\frac{2X_N - N}{\sqrt{N}}\right) \approx N \cdot \frac{1}{2}\left(1 - \frac{1}{2}\right) \cdot \frac{4}{N} = 1\,. \tag{5}$$

For the variance of Y we obtain

$$\mathrm{Var}(Y) = \sigma^2 T\,. \tag{6}$$

Therefore, as the time to maturity T is fixed, we can now take the limit $N \to \infty$ corresponding to $\delta t \to 0$. In this limit, the binomial distribution tends to a normal distribution and we can write S_T therefore as

$$S_T \approx S_0\,\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}, \qquad Z \sim N(0,1)\,. \tag{7}$$

We are now ready to write the option price as an expectation value: For a European call option $X = (S_T - K)^+$ the expectation of the discounted claim is written as

$$\begin{aligned} V &= \mathbb{E}\left(\mathrm{e}^{-rT}X\right) = \mathbb{E}\left((\mathrm{e}^{-rT}S_T - K\mathrm{e}^{-rT})^+\right) \\ &= \mathbb{E}\left((S_0\,\mathrm{e}^{\sigma\sqrt{T}Z-\sigma^2T/2} - K\mathrm{e}^{-rT})^+\right) \end{aligned}$$

Rewriting the integral using the cumulative normal distribution: The above expectation value can be rewritten by using the cumulative normal distribution Φ defined as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,dy\,. \tag{8}$$

Theorem. (Black-Scholes Option Pricing) *The price V of a European call option with strike price K is given by*

$$\begin{aligned} V &= \mathbb{E}\left((S_0\,\mathrm{e}^{\sigma\sqrt{T}Z-\sigma^2T/2} - K\mathrm{e}^{-rT})^+\right) \\ &= S_0\Phi(d_1) - K\,\mathrm{e}^{-rT}\,\Phi(d_2) \end{aligned} \tag{9}$$

with

$$d_1 = \frac{\ln(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\,. \tag{10}$$

Proof. In order to compute the above integral, let's first write

$$V = \int_{-\infty}^{\infty} p(z) \left(S_0 \, \mathrm{e}^{\sigma\sqrt{T} z - \sigma^2 T/2} - K \mathrm{e}^{-rT} \right)^+ \, dz \,, \qquad p(z) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-z^2/2} \,.$$

Now we note that

$$\left(S_0 \, \mathrm{e}^{\sigma\sqrt{T} z - \sigma^2 T/2} - K \mathrm{e}^{-rT} \right)^+ = \max \left(S_0 \, \mathrm{e}^{\sigma\sqrt{T} z - \sigma^2 T/2} - K \mathrm{e}^{-rT}, 0 \right) \,.$$

For values of z that are negative and large in absolute value, the contribution will be zero as the first term will be negative. However, as z increases, we see, that there is a 'critical' $z = z_c$ such that for $z > z_c$, there will be non-zero contributions to the integral. This z_c is determined by the equation

$$S_0 \, \mathrm{e}^{\sigma\sqrt{T} z_c - \sigma^2 T/2} = K \mathrm{e}^{-rT} \tag{11}$$

Solving this equation for z_c yields after a couple of lines of algebra

$$z_c = \frac{\ln(K/S_0) + (\sigma^2/2 - r)T}{\sigma\sqrt{T}} \,. \tag{12}$$

This allows us to rewrite the integral for the option price V by

$$V = S_0 \int_{z_c}^{\infty} \mathrm{e}^{\sigma\sqrt{T} z - \sigma^2 T/2} p(z) \, dz - K \mathrm{e}^{-rT} \int_{z_c}^{\infty} p(z) dz \tag{13}$$

For the second integral, we find immediately ($\tilde{z} = -z$)

$$\begin{aligned} \int_{z_c}^{\infty} p(z) dz &= \int_{-\infty}^{-z_c} p(\tilde{z}) d\tilde{z} = \Phi(-z_c) \\ &= \Phi \left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) = \Phi(d_2) \end{aligned}$$

The first integral is left as an exercise. $\square$

Review of the Taylor expansion

Basic Taylor expansions: A fundamental idea of calculus consists in approximating a function - locally in the neighborhood of the point $(x_0, f(x_0))$ - if at all possible, through its tangent:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)\,.$$

What if we wanted to use a higher-order polynomial, say a quadratic function to improve (locally) this approximation? How would we find the coefficients c_0, c_1, c_2 in

$$f(x) \approx c_0 + c_1(x - x_0) + c_2(x - x_0)^2$$

Extending the basic idea of the tangent line, we would try to match higher derivatives (here the second derivative) at x_0. Computing the first two derivatives, we find

$$\begin{aligned}
f(x) &\approx c_0 + c_1(x - x_0) + c_2(x - x_0)^2 \\
f'(x) &\approx c_1 + 2c_2(x - x_0) \\
f''(x) &\approx 2c_2
\end{aligned}$$

and evaluating these approximations at x_0, we find the following equations for the coefficients:

$$f(x_0) = c_0, \qquad f'(x_0) = c_1, \qquad f''(x_0) = 2c_2\,.$$

Here, the first two coefficients reproduce the tangent-line approximation, and the following (new) coefficient c_2 for a parabolic approximation is given by

$$c_2 = \frac{1}{2} f''(x_0)\,.$$

We can extend this idea easily to a polynomial of degree n. Assume that we would like an approximation of the form

$$f(x) \approx \sum_{k=0}^{n} c_k (x - x_0)^k\,. \tag{14}$$

As before, we can compute derivatives in order to match them at x_0. We have

$$\begin{aligned}
f(x) &\approx c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + ... + c_n(x - x_0)^n\,, \\
f'(x) &\approx c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + ... + nc_n(x - x_0)^{n-1}\,, \\
f''(x) &\approx 2c_2 + 3 \cdot 2c_3(x - x_0) + 4 \cdot 3c_4(x - x_0)^2 + ... + n(n-1)c_n(x - x_0)^{n-2}\,.
\end{aligned}$$

From here we find immediately the formula for the k-th coefficient c_k:

$$c_k = \frac{1}{k!} f^{(k)}(x_0)\,, \tag{15}$$

and the n-th order Taylor polynomial approximating the function f is therefore given as

$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k\,. \tag{16}$$

Note that, if the corresponding Taylor series converges, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k\,. \tag{17}$$

A few examples: Consider $x_0 = 0$ as the expansion point.

1. $f(x) = \mathrm{e}^x$. Here $f^{(k)}(0) = 1$. Therefore, we find directly

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

2. $f(x) = \cos(x)$. Now $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1, ...$ Thus we have even terms and alternating signs:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k}\,.$$

3. $f(x) = \sin(x)$. Now $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0, ...$ Thus we have odd terms and alternating signs:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k x^{2k+1}\,.$$

It is noteworthy that we can derive from the above *Euler's formula*:

$$\mathrm{e}^{ix} = \cos(x) + i\sin(x) \tag{18}$$

We can use the Taylor polynomial to find approximations for small arguments. Consider $\epsilon \ll 1$. Then clearly $\epsilon \gg \epsilon^2 \gg \epsilon^3 \gg .....$ Therefore, writing

$$\mathrm{e}^{\epsilon} \approx 1 + \epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{6}\epsilon^3 + ... \tag{19}$$

we find that the terms involving high powers of ϵ are less important than the small powers of ϵ. Sometimes, we refer to terms for $\mathcal{O}(\epsilon^k)$ to collect all terms that involve the k-th power of ϵ. For reasons that will become clear later, when manipulating stochastic equations, we will have to deal with expansions that involve $\sqrt{\epsilon} = \epsilon^{1/2}$. This makes the bookkeeping a little tricky. Note that, for $\epsilon < 1$ we have $\sqrt{\epsilon} > \epsilon$, and this inequality becomes more and more 'dramatic' as ϵ approaches zero.

Let's practice the use of these expansions using an example that we will need later in our first derivation of the Black-Scholes formula. First, write

$$\mathrm{e}^{r\epsilon} \approx 1 + r\epsilon + \frac{1}{2}r^2\epsilon^2 + ... \tag{20}$$

Here, we have written all relevant terms including terms of order $\mathcal{O}(\epsilon^2)$. Expanding a different term $\mathrm{e}^{\sigma\sqrt{\epsilon}}$ we see that we need many more terms before we arrive at $\mathcal{O}(\epsilon^2)$:

$$\mathrm{e}^{\sigma\sqrt{\epsilon}} = 1 + \sigma\sqrt{\epsilon} + \frac{1}{2}\sigma^2\epsilon + \frac{1}{3!}\sigma^3\epsilon^{3/2} + \frac{1}{4!}\sigma^4\epsilon^2 + ... \tag{21}$$

In a similar way we can expand a term $\mathrm{e}^{-\sigma\sqrt{\epsilon}}$, realizing that the sign only affects odd powers:

$$\mathrm{e}^{-\sigma\sqrt{\epsilon}} = 1 - \sigma\sqrt{\epsilon} + \frac{1}{2}\sigma^2\epsilon - \frac{1}{3!}\sigma^3\epsilon^{3/2} + \frac{1}{4!}\sigma^4\epsilon^2 + ... \tag{22}$$

Most of the time, we are interested in the terms up $\mathcal{O}(\epsilon)$, but if what we are expanding involves terms of $\sqrt{\epsilon}$, we usually have to use a 2nd-order Taylor expansion. In this context, this seems a rather trivial observation, but we will see later that this is the key to stochastic calculus. We are now ready to derive a result that we need later: First observe that, up to $\mathcal{O}(\epsilon)$, we have

$$\mathrm{e}^{\sigma\sqrt{\epsilon}} - \mathrm{e}^{-\sigma\sqrt{\epsilon}} \approx 2\sigma\sqrt{\epsilon} \tag{23}$$

The precise expansion that we will need later is the following:

$$\begin{aligned}
\frac{\mathrm{e}^{r\epsilon} - \mathrm{e}^{\mu\epsilon-\sigma\sqrt{\epsilon}}}{\mathrm{e}^{\mu\epsilon+\sigma\sqrt{\epsilon}} - \mathrm{e}^{\mu\epsilon-\sigma\sqrt{\epsilon}}} &= \frac{\mathrm{e}^{(r-\mu)\epsilon} - \mathrm{e}^{-\sigma\sqrt{\epsilon}}}{\mathrm{e}^{\sigma\sqrt{\epsilon}} - \mathrm{e}^{-\sigma\sqrt{\epsilon}}} \\
&\approx \frac{1 + (r-\mu)\epsilon - (1 - \sigma\sqrt{\epsilon} + \frac{\sigma^2}{2}\epsilon)}{2\sigma\sqrt{\epsilon}} \\
&= \frac{\left(r - \mu - \frac{\sigma^2}{2}\right)\epsilon + \sigma\sqrt{\epsilon}}{2\sigma\sqrt{\epsilon}} \\
&= \frac{1}{2}\left(1 - \sqrt{\epsilon}\left(\frac{\mu + \frac{\sigma^2}{2} - r}{\sigma}\right)\right)
\end{aligned}$$

Note that, although we kept at the beginning terms up to $\mathcal{O}(\epsilon)$, due to cancellations, our approximation is only correct up to $\mathcal{O}(\sqrt{\epsilon})$, but it turns out that (in the case where will apply this expansion) this is, fortunately, sufficient.

Taylor expansions in two dimensions: It is easy to extend the idea of a Taylor expansion to higher dimensions. In two dimensions, we find as an expansion in the neighborhood of a point (x_0, y_0):

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x-x_0)(y-y_0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y-y_0)^2 \,, \end{aligned}$$

where all derivatives are evaluated at the point (x_0, y_0). Therefore, for small deviations from (x_0, y_0) we can write

$$\begin{aligned} f(x_0+\epsilon, y_0+\delta) &\approx f(x_0,y_0) + \frac{\partial f}{\partial x}\epsilon + \frac{\partial f}{\partial y}\delta \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon^2 + \frac{\partial^2 f}{\partial x \partial y}\epsilon\delta + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\delta^2 \,, \end{aligned}$$

If the argument involves roots, we proceed as before. Consider for example $f(x,y) = xy$. Then we find immediately

$$f(x_0+\sqrt{\epsilon}, y_0+\sqrt{\delta}) \approx f(x_0,y_0) + y_0\sqrt{\epsilon} + x_0\sqrt{\delta} + \sqrt{\epsilon}\sqrt{\delta} \qquad (24)$$