

Martingales

We will now analyze the mathematical structure behind the option pricing. First, we need several basic definitions, then we will see how they can be applied in a financial context.

Stochastic Process: A stochastic process Y is a collection of random variables, usually we write $Y = (Y_0, Y_1, Y_2, \dots)$. We interpret the index as the number of the time-tick, and, at the moment, we only consider discrete stochastic processes. Usually, we assume that the value of Y_0 is known, and the other Y_i for $i > 0$ are random variables with several possible values. An example is the stock process S given by the above tree. Here, for example, $S_0 = 100$ and S_2 can take the values $\{60, 100, 140\}$.

Filtration: A filtration \mathcal{F}_i is a history in time up to the time i . In the case of the stock process, a particular path in the tree corresponds to such a history. There are different ways to identify the history, for example by listing the corresponding nodes or stock values. Alternatively, we can also say, for instance, that we are looking at the filtration $\mathcal{F}_2 = (u, u)$ where 'u' stands for an up-movement of the stock process. We could also say $\mathcal{F}_2 = (100, 120, 140)$ meaning that \mathcal{F}_2 is the path of the stock where $S_0 = 100$, $S_1 = 120$, and $S_2 = 140$.

Conditional expectation: Given a filtration \mathcal{F}_i up to the time i , we can compute expectations with respect to the remaining part of the tree, taking into account the nodes that are still accessible given the history \mathcal{F}_i . For the above history (u, u) , the stock will be at the node with the value 140. Now only the two remaining values 120 and 160 are accessible. Therefore, we can compute the expectation of S_3 conditioned on the history $\mathcal{F}_2 = (u, u)$. We write $\mathbb{E}_{\mathbb{Q}}(S_3 | \mathcal{F}_2 = (u, u))$. In our case, we see that

$$\mathbb{E}_{\mathbb{Q}}(S_3 | \mathcal{F}_2 = (u, u)) = \frac{1}{2} \cdot 160 + \frac{1}{2} \cdot 120 = 140.$$

Martingales: It is easy to check that, for the above example, we have

$$\mathbb{E}_{\mathbb{Q}}(S_3 | \mathcal{F}_2) = S_2$$

for *any* history \mathcal{F}_2 . Or, even more general $\mathbb{E}_{\mathbb{Q}}(S_j | \mathcal{F}_i) = S_i$ for $j \geq i$. This property is called the martingale property and a stochastic process Y with the property

$$\mathbb{E}_{\mathbb{Q}}(Y_j | \mathcal{F}_i) = Y_i \tag{1}$$

for $j \geq i$ is called a \mathbb{Q} -martingale. Why is the stock process considered above a \mathbb{Q} -martingale? The answer is extraordinarily simple: We constructed the

measure \mathbb{Q} such that S has exactly this martingale property! Remember that we used the formula

$$q = \frac{s_{now} - s_d}{s_u - s_d}, \quad s_{now} = qs_u + (1 - q)s_d.$$

The latter equation ensures that

$$S_k = \mathbb{E}_{\mathbb{Q}}(S_{k+1} | \mathcal{F}_k)$$

such that the martingale property holds at each node of the tree. It is then easy to see, e.g. by induction, that the martingale property holds for the entire tree.

The claim: Let's now consider the claim X , for example the European call $X = (S_3 - K)^+$. Here things are slightly more complicated as X is, at first, only defined on the end nodes of the tree. When filling the option tree, working backwards, we constructed a new stochastic process Y , such that $Y_3 = X$ at the end nodes. At each step, we computed

$$f_{now} = qf_u + (1 - q)f_d.$$

In other words, we used in fact the conditional expectation operator in order to construct Y , hence

$$Y_i = \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_i) \tag{2}$$

For instance, for the filtration $\mathcal{F}_2 = (u, u)$, we have

$$f_{now} = \mathbb{E}_{\mathbb{Q}}((S_3 - K)^+ | \mathcal{F}_2 = (u, u)) = \frac{1}{2} \cdot 60 + \frac{1}{2} \cdot 20 = 40.$$

In this way, Y is again a martingale, more precisely a \mathbb{Q} -martingale as we used the measure \mathbb{Q} in the conditional expectation operator that defines Y .

Binomial Representation Theorem

Construction strategies: So far, the mathematical view of option pricing consists in the following steps (remember, for now, we assume that the interest rate $r = 0$):

1. For the given stock process S , construct a measure \mathbb{Q} , such that S is a \mathbb{Q} -martingale.
2. Convert a claim X , that is defined at the end nodes of the tree, into a stochastic process Y defined on the same tree as S using the conditional expectation operator

$$Y_i = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i).$$

3. The option price is $Y_0 = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0) = \mathbb{E}_{\mathbb{Q}}(X)$.

The final step is to clarify what the replication strategy (the stock and bond holdings in order to hedge the claim) means in mathematical terms. The appropriate interpretation is the following: We have two \mathbb{Q} -martingales on the same tree, and, in such a situation, it can be shown that one martingale can be constructed from the other martingale in a *previsible* way, meaning that we always know one step ahead which ϕ to choose (which is important for our hedging strategy). To formulate this more precisely, we prove the following theorem:

Theorem. (Binomial Representation Theorem) *Assume that S is a \mathbb{Q} -martingale and V is another \mathbb{Q} -martingale on the same tree. Then there exists a previsible process ϕ such that*

$$V_n = V_0 + \sum_{j=0}^{n-1} \phi_{j+1}(S_{j+1} - S_j) \quad (3)$$

Proof. Consider a step from i to $i+1$ where $S_i = s_{now}$ can go to $S_{i+1} = s_u$ or $S_{i+1} = s_d$ and $V_i = f_{now}$ can go to $V_{i+1} = f_u$ or $V_{i+1} = f_d$. Clearly, we can find ϕ_{i+1} and k_{i+1} such that

$$\begin{aligned} f_u - f_{now} &= \phi_{i+1}(s_u - s_{now}) + k_{i+1} \\ f_d - f_{now} &= \phi_{i+1}(s_d - s_{now}) + k_{i+1} \end{aligned}$$

In particular we know at time i the value of ϕ_{i+1} to be

$$\phi_{i+1} = \frac{f_u - f_d}{s_u - s_d}$$

To prove the formula

$$V_{i+1} - V_i = \phi_{i+1}(S_{i+1} - S_i),$$

we need to show that $k_{i+1} = 0$ and to do so, we will make use of the assumptions that the processes S and V are \mathbb{Q} -martingales: Since k_{i+1} is known at time i we have

$$\begin{aligned} k_{i+1} &= \mathbb{E}_{\mathbb{Q}}(k_{i+1}|\mathcal{F}_i) \\ &= \mathbb{E}_{\mathbb{Q}}(V_{i+1} - V_i|\mathcal{F}_i) - \phi_{i+1}\mathbb{E}_{\mathbb{Q}}(S_{i+1} - S_i|\mathcal{F}_i) \\ &= 0. \end{aligned}$$

The theorem then follows by induction. \square

Replication with non-zero interest rates: For the case $r \neq 0$, it can be shown that we need to make only minor modifications to the above application of the binomial representation theorem: Define a process Z given by $Z_i = B_i^{-1}S_i$ (the *discounted* stock process), and choose the measure \mathbb{Q} such that Z is a \mathbb{Q} -martingale. Then define a process E using the conditional expectation operator

$$E_i = \mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_i) \tag{4}$$

and apply the binomial representation theorem such that

$$E_n = E_0 + \sum_{j=0}^{n-1} \phi_{j+1}(Z_{j+1} - Z_j) \tag{5}$$