Binomial pricing model with non-zero interest rates

Assume now that we have non-zero interest rates, such that ψ units of the cash bond B_0 will have the value

$$\psi B_0 \mathrm{e}^{r\delta t}$$

after a time δt. Again, we consider the basic binomial pricing situation where the stock can have two values after the time-tick δt, together with a derivative that also can have two values.

$$\begin{tikzcd} & s_u \\ s_{now} \arrow[ur] \arrow[dr] & \\ & s_d \end{tikzcd} \qquad \begin{tikzcd} & f_u \\ f_{now}=??? \arrow[ur] \arrow[dr] & \\ & f_d \end{tikzcd}$$

Again, we will set up a portfolio V at the beginning to replicate the claim, and the money we need to create this portfolio will correspond to the price that we charge for the derivative, hence $V_{now} = f_{now}$. Consider

$$V = \phi S + \psi B$$

Before the clock ticks, this portfolio has the value

$$V_{now} = f_{now} = \phi s_{now} + \psi B_0 \tag{1}$$

and, after the time-tick, we need to satisfy the equations

$$\begin{aligned} f_u &= \phi s_u + \psi B_0 \mathrm{e}^{r\delta t} \\ f_d &= \phi s_d + \psi B_0 \mathrm{e}^{r\delta t} \end{aligned}$$

and, again, we find that

$$\phi = \frac{f_u - f_d}{s_u - s_d}\,.$$

Note that this is the same formula as in the case of zero interest rates: In order to replicate the claim, a certain number of shares of the stock are necessary: The amount of stock to hold is given by the ratio of the difference of the claim and the difference of the stock given by the two possible scenarios. In the spirit of replicating the claim by the appropriate stock holding, we can also write equivalently

$$f_u - f_d = \phi(s_u - s_d)\,,$$

which is, therefore, trivial to remember. In order to proceed with the calculation of the initial price of the claim (or the initial portfolio), we still need to find the holding ψ of the cash bond. Going back to the above system of equations, if we multiply the first equation by s_d and the second equation by s_u and subtract the first equation from the second equation, we obtain directly

$$\psi = B_0^{-1} \mathrm{e}^{-r\delta t} \left(\frac{f_d s_u - f_u s_d}{s_u - s_d} \right) \tag{2}$$

and, putting everything together, we find the formula for the price of the initial portfolio (and hence the value of the claim before the time-tick) as

$$V = \frac{f_u - f_d}{s_u - s_d} s_{now} + \mathrm{e}^{-r\delta t} \left(\frac{f_d s_u - f_u s_d}{s_u - s_d} \right) \tag{3}$$

Again, it is important to see that this formula is entirely independent of any "real-world" probabilities that one might associate with the event of the stock S going up to s_u or going down to s_d. The price of the claim is determined by the idea of setting up a risk-less portfolio that replicates the claim, nothing else, a mechanism that is independent of the real-world probabilities. The estimate of the real-world probabilities might play a role, whether a buyer finds the claim attractive: Consider a call option that is attractive if the buyer believes that the stock is likely to go up: Even if most people feel, that there is a 90% chance of the stock to go up, the seller of the option will still price it independent of this probability: The option price is enforced by the requirement of setting up a risk-less portfolio.
Risk-neutral probability measure: The formula for the claim on a binomial branch (3) can be rewritten in a form that is much simpler to remember: First, define q as

$$q = \frac{\mathrm{e}^{r\delta t} s_{now} - s_d}{s_u - s_d} . \tag{4}$$

It can be shown that $0 < q < 1$. With this q, we can calculate the price of the initial portfolio as

$$V = \mathrm{e}^{-r\delta t} \left(q f_u + (1 - q) f_d \right) . \tag{5}$$

In the interest-free case $r = 0$, this formula has the form of an expectation for a binomial branch with the probability q.

$$V = f_{now} = q f_u + (1 - q) f_d \tag{6}$$

Interestingly, in that case, the q is defined by (4) in a way that

$$s_{now} = q s_u + (1 - q) s_d \tag{7}$$

In this way, the pricing formula is simple to remember (at least for the case $r = 0$): First find the *risk-neutral* probability q defined such that the current stock value is the expectation of two possible future stock values and then use this probability to compute the initial price of the claim by taking the expectation of the two possible claim values.

$$\begin{tikzcd}[column sep=large] & s_u \\ s_{now} \arrow[ur, "q"] \arrow[dr, "1\text{-}q"'] & \\ & s_d \end{tikzcd} \qquad \begin{tikzcd}[column sep=large] & f_u \\ f_{now} \arrow[ur, "q"] \arrow[dr, "1\text{-}q"'] & \\ & f_d \end{tikzcd}$$

In this language, we can use all the tools from probability theory - knowing that q is not a "real-world" probability, but the risk-neutral probability. Returning to the example of a bet that pays \$1 if a stock goes up (and the stock, priced at \$1 now can take the values \$2 and \$0.50 after the time-tick), we find that

$$q = \frac{1 - 0.5}{2 - 0.5} = \frac{1}{3}, \qquad V = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 0 = \frac{1}{3}\,.$$

Example for a call option: As a second example, take the above case of the a stock that is worth now \$100 and can go up to \$150 or down to \$90. Consider a strike price $K = \$120$ and a time-tick of one year with $r = 5\%$. We then compute q as

$$q = \frac{\mathrm{e}^{r\delta t} s_{now} - s_d}{s_u - s_d} = \frac{100 \cdot \mathrm{e}^{0.05} - 90}{150 - 90} = 0.2521$$

and find as option value

$$V = \mathrm{e}^{-0.05}\left(q \cdot 30 + (1 - q) \cdot 0\right) = 7.19$$

Binomial Trees

From branch to tree: With the basic pricing formula for a branch on hand, we can extend the pricing to trees easily. Simply apply the branch formula to each branch of the tree. Let's illustrate this idea using a simple example. Consider three time-ticks, and a stock with a value of 100 at the beginning that can go up or down 20 at each time-tick. Then, the corresponding stock process is given by the following tree:

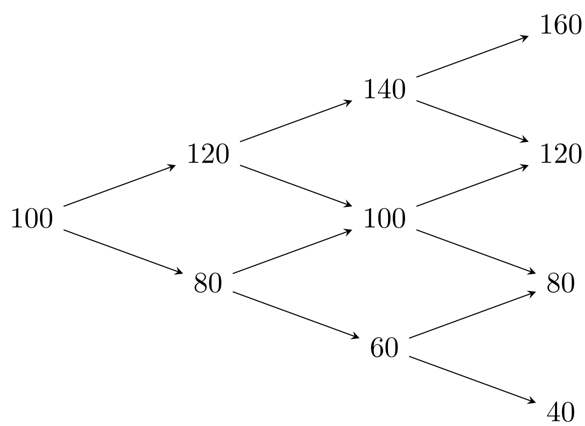

Figure 1: Binomial tree of the stock process

Consider, for instance, the branch on the top right corner. Here, $s_u = 160$, $s_d = 120$, and $s_{now} = 140$. Since the strike price of the option is $K = 100$, we know that, at the part of the tree, we have $f_u = 60$ and $f_d = 20$. From the stock values, we find immediately the risk-neutral probability q as

$$q = \frac{s_{now} - s_d}{s_u - s_d} = \frac{140 - 120}{160 - 120} = \frac{1}{2}$$

and, therefore, $f_{now} = qf_u + (1 - q)f_d = 40$. It is easy to see that, in this particular case, we have $q = 1/2$ on the entire tree. Since the option value is known at the end nodes of the tree, we can work backward and obtain the option price at the beginning of the tree (here 15). Figure shows the corresponding option tree. Working backward is a good idea as we need to set up a portfolio that replicates the value of the claim (here the option) at all nodes of the tree. Let us know see how the hedging works for a particular path:

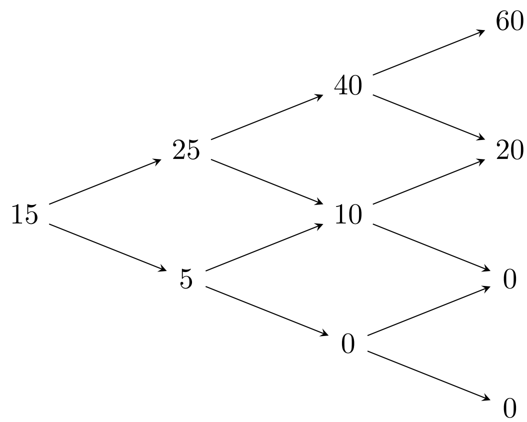

Figure 2: Binomial tree of the option process

Time $i = 0$ We are given 15 for the option. To set up the portfolio, we need to know how much of the stock is required. As before, we write the portfolio as

$$V = \phi S + \psi B$$

At time $i = 0$, we find the amount of stock we need to hold to be risk-neutral to be

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{25 - 5}{120 - 80} = \frac{1}{2}$$

As the price of the stock at the beginning of the tree is $S_0 = 100$, the cost is $0.5 \cdot 100 = 50$ and we are given 15 for writing the option. Hence we need to borrow 35 (and we assume, for simplicity, at the moment that the interest rate $r = 0$). Therefore, our bond holding is now -35. To summarize, at time $i = 0$, before the first time-tick, our holdings are $(\phi, \psi) = (0.5, -35)$.

Time $i = 1$ Assume that the stock goes up. Now we need to see whether we need to adjust our holdings of stock and bond. First, we need to compute the new ϕ. The ϕ necessary to be risk-neutral at this node of the tree is computed by looking at the next step ahead, hence

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{40 - 10}{140 - 100} = \frac{3}{4}$$

Since we are already holding 0.5 shares in our portfolio, we need to buy only 0.25 shares in addition. Now, the stock is worth $S_1 = 120$, hence the cost of the additional shares is $0.25 \cdot 120 = 30$, which brings our debt to 65. To summarize, at time $i = 1$, at the node where $S_1 = 120$, our holdings are $(\phi, \psi) = (0.75, -65)$.

Time $i = 2$ The new ϕ is now

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{60 - 20}{160 - 120} = 1$$

Note, that at this node, the option is in the money, for sure and will be exercised at the next time-tick. Considering our portfolio, we are already holding 3/4 of shares, so we need to buy 1/4 in addition. The price of the stock is now $S_2 = 140$, so the cost of this additional 0.25 is $0.25 \cdot 140 = 35$ which brings our debts to 100. To summarize, at time $i = 2$, at the node where $S_1 = 140$, our holdings are $(\phi, \psi) = (1, -100)$.

Time $i = 3$ The clock ticks one more time and the holder needs to decide whether to exercise the option. The bank, on the other hand, has created a portfolio whose value replicates the value of the claim (the option) such that the bank is risk-neutral. In our case, we see that if the stock goes up one more time, to $S_3 = 160$, the option in worth $S_3 - K = 60$. The portfolio has the same value:

$$V = \phi S + \psi B = 1 \cdot 160 - 100 \cdot 1 = 60$$

If, on the other hand, the stock happens to go down at the last step, hence $S_3 = 120$, the option is worth 20. Obviously, in this case, the portfolio has the value 20 as well:

$$V = \phi S + \psi B = 1 \cdot 120 - 100 \cdot 1 = 20$$

Either way, the portfolio has the same value as the option at maturity, hence the bank remains risk-neutral. Note that, in order to maintain risk-neutrality, the bank had to adjust the holdings (ϕ, ψ) in the portfolio according to the stock movements. This is called a *dynamic hedging strategy*. Also, the same idea works for any stock path (but the actual values of (ϕ, ψ) differ usually for different paths. However, no matter which path the stock takes, the initial money paid for the option (here the 15 dollars), is *always* sufficient to set up a risk-free portfolio. Hence, the hedging strategy is self-financing. Of course, we might need to borrow money to buy part of the stock, but we will always be able to pay back our debts at the end.