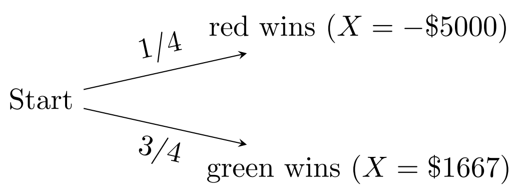
The parable of the bookmaker

Consider a race between two horses (”red” and ”green”). Assume that the bookmaker estimates the chances of ”red” to win as 25% (and hence the chances of ”green” to win are 75%). This corresponds to 3-1 against ”red” (or 1-3 on ”green”). Let’s assume that \$5,000 are bet on ”red”, and \$10,000 on ”green”. We define a random variable X for the profit (or loss) of the bookmaker after the race. If ”red” wins, he needs to pay \$3·5,000, but keeps the \$10,000, so X is -\$5,000. If ”red” loses, ”green” wins, and the bookmaker has to pay \$10,000/3, but keeps the \$5,000.

So, in this case, X takes the value \$5,000/3 $\approx$ \$1667. In summary, the bookmaker might win or lose money. This means that there is a risk for the bookmaker - equivalent to himself was betting on the race.
We can cast this in terms of probabilities: let $p = 1/4$ be the probability that ”red” wins. The diagram below illustrates the situation:

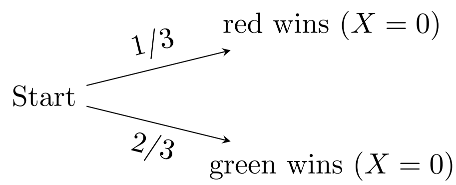


Clearly, we have $\mathbb{E}(X) = \$0$, but this is only an average taken over many (theoretical) realizations of X.

However, things do not have to be that way. The risk clearly depends on the way the bookmaker is quoting the odds. Therefore, we might ask the question: Is there a way to quote odds such that the bookmaker will remain *risk-neutral*? This might seem odd at first, but - from the point of view of the bookmaker, this is the most reasonable position to take. Of course, he will take a commission for his services and make a living in that way - without any risk related to the random outcome of the race.

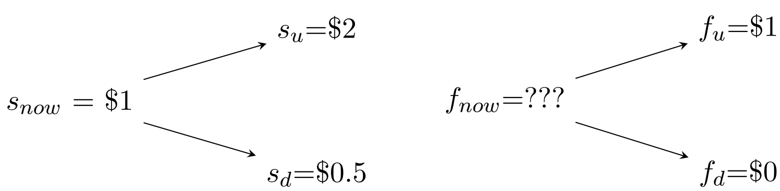
Indeed, this is possible: If the bookmaker quotes the odds as 2-1 against ”red”, he will be risk-neutral: If ”red” wins, he needs to pay \$2·5,000, but keeps the \$10,000. If ”red” loses the race, he needs to pay \$10,000/2, but keeps the \$5,000. In either case, the bookmaker breaks even, there is no risk in selling the bets.

Note that these odds only depend on the sums of money that were bet on the horses - not on the real-world probabilities of the horses to win the race. In fact, such real-world probabilities are difficult to estimate, but in



quoting the odds for the race they do not play any role for a bookmaker who intends to remain risk-neutral.

The situation is similar in finance when dealing with so-called *derivative* which are contract that are derived from fundamental assets. Consider, for instance a stock that is worth \$1. After a time δt, the stock can either go up to \$2 or go down to \$0.5. What is the price of a bet that pays \$1 if the stock goes up?



The main idea is that the seller of the bet can invest in the stock to hedge the claim and this possibility gives him a chance to sell the bet and still stay risk-neutral. All that he needs to do is to set up a portfolio that will have the worth of the claim after the time δt. Let's denote the value of the bet (after the time-tick) by $f_u p = \$1$, if the stock goes up and $f_d = \$0$, if the stock goes down.

Consider a portfolio of ϕ units of stock and ψ units of a cash bond. For simplicity, we assume that the interest rate is zero. At the beginning, before the time-tick δt, the worth of the portfolio is

$$V = \phi S + \psi B\,.$$

Here, S is the current stock price (in our case \$ 1) and B - as we are working in dollars, we set $B = \$1$. When the clock ticks, the value of ψB will not change (since we assumed that the interest rate is zero), but the value of ϕS will change, since the value S after the time-tick δt is random. If the stock goes up, we will have $S = s_u = \$2$ and if the stock goes down, we will have $S = s_d = \$0.5$. If you are selling the bet and if you want to be risk-neutral, you will tri to adjust the portfolio (hence ϕ and ψ) such that V will have the value of \$1 if the stock goes up and \$0 if the stock goes down (to mimic the claim). It is easy to figure out what ϕ and ψ should be:

$$
\begin{aligned}
V_u &= \phi s_u + \psi B = f_u = 1 \\
V_d &= \phi s_d + \psi B = f_d = 0
\end{aligned}
$$

This is an equation with two unknowns, and clearly we have

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{1-0}{2-0.5} = \frac{2}{3} \tag{1}$$

and, from either equation, we find $\psi = -1/3$. This means that, in oder to set up a risk-free portfolio that mimics the bet (claim), one needs

$$V = \frac{2}{3} \cdot \$1 - \frac{1}{3} \cdot \$1 = \$0.33 \,.$$

And this is exactly the price (or worth) of the bet that the seller will ask from the buyer.

Basics of financial markets, derivatives

Stock and Bond: Our basic financial market consists of two types of assets: stocks and bonds. The stock is random, meaning that we cannot predict its value for future times. We will see later that exponential Brownian motion is a basic model and write

$$S_t = S_0 \, \mathrm{e}^{\mu t + \sigma W_t} \,.$$

The other asset, the cash bond, is deterministic. If we assume an interest $0 \le r$ and compound continuously, we find that the value of the bond at a future time t is known:

$$B_t = B_0 \, \mathrm{e}^{rt} = B_0 \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt}$$

Most of the time, we will set $B_0 = 1$ (think of it as \$1 at time $t = 0$).
Basic assumptions: In our analysis, we usually assume the following:

- no transaction costs
- no tax
- unlimited borrowing/short-selling

- fixed interest rate, same rate if you borrow or lend
- all assets can be split
- no arbitrage

Derivatives: If an asset is derived from a basic asset, we call it a derivative. Options are important examples. The buyer of the option acquires the right (but has no obligation) to do something (usually to buy or to sell an asset for an agreed price) at a future time.

- *Call option:* gives the holder of the option the right to buy a stock for a price K.
- *Put option:* gives the holder of the option the right to sell a stock for a price K.

In both cases, we call K the *strike price*. The corresponding pay-offs are

- Pay-off of a call option: $(S_T - K)^+ = \max(S_T - K, 0)$.
- Pay-off of a put option: $(K - S_T)^+ = \max(K - S_T, 0)$.

Moreover, we distinguish between *European* and *American* options:

- *European option:* can only be exercised at the expiration date.
- *American option:* can be exercised at expiration date or any time before expiration date

In the following, for simplicity, we will focus on European options. Example: Consider a European call of a stock that is worth now $S_0 = \$100$ with strike price $K = \$120$ and a maturity of $T = 2$ years. If, at expiration, the stock is worth $S_T = \$150$, the worth of the call is the difference, hence \$30: The holder of the option will exercise the option, hence buy a stock for $K = \$120$ and then sell it at the current value of \$150. If, on the other hand, the stock happens to be worth $S_T = \$90$, the option will expire worthless (and not exercised, as nobody would pay $K = \$120$ for a stock that one can buy for \$90).

Arbitrage

Why is it so important to price options correctly? Consider, for example, a stock is worth \$100 now, a bond worth \$100 as well. Assume that the stock could go up or down \$20 in one year (so $s_u = \$120$ and $s_d = \$80$), and that the bond will be worth \$110. Assume that a bank offers a European call, $K = \$100$ for \$10. What would you do? Here is a smart idea: Buy 2/5 of the bond, one call option, sell 1/2 of the stock. The cost to set up this portfolio is

$$V = \frac{2}{5} \cdot 100 + 10 - \frac{1}{2} \cdot 100 = 0\,.$$

So, you can set up this portfolio for free. What will happen in one year? If the stock goes up, the call will be worth \$20 and, therefore,

$$V = \frac{2}{5} \cdot 110 + 20 - \frac{1}{2} \cdot 120 = 44 + 20 - 60 = 4\,.$$

If, on the other hand, the stock goes down, we find

$$V = \frac{2}{5} \cdot 110 + 0 - \frac{1}{2} \cdot 80 = 44 + 0 - 40 = 4\,.$$

We would have found a way to make money for free! Such arbitrage opportunities should not exist in a market that is in equilibrium - and a correct (risk-free inspired) pricing of options is essential for this.