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Frequently visited sets for random walks

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Abstract

We study the occupation measure of various sets for a symmetric transient random walk in Z^d with finite variances. Let $\mu_n^X(A)$ denote the occupation time of the set A up to time n. It is shown that $\sup_{x\in Z^d} \mu_n^X(x+A)/\log n$ tends to a finite limit as $n\to\infty$. The limit is expressed in terms of the largest eigenvalue of a matrix involving the Green function of X restricted to the set A. Some examples are discussed and the connection to similar results for Brownian motion is given.

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1. Introduction

Let $X_n,\ n = 0, 1, \ldots$ be a symmetric transient random walk in Z^d ($d \geqslant 3$). We will always assume that $X_n,\ n = 0, 1, \ldots$ is not supported on any subgroup strictly smaller than Z^d. We denote by μ_n^X its *occupation measure*:

$$\mu_n^X(A) = \sum_{j=0}^{n} \mathbf{1}_A(X_j)$$

for all sets $A \subseteq Z^d$. Let $q_n(x) = \mathbf{P}(X_n = x)$. As usual, we let

$$G(x) = \sum_{k=0}^{\infty} q_k(x) \tag{1.1}$$

denote the Green's function for $\{X_n\}$. For any finite $A \subseteq Z^d$ let Λ_A denote the largest eigenvalue of the $|A| \times |A|$ matrix

$$G_A(x, y) = G(x - y), \quad x, y \in A. \tag{1.2}$$

Theorem 1.1. *If X has finite second moments then*

$$\lim_{n\to\infty} \sup_{x\in Z^d} \frac{\mu_n^X(x + A)}{\log n} = -1/\log(1 - 1/\Lambda_A) \quad \text{a.s.} \tag{1.3}$$

and

$$\lim_{n\to\infty} \sup_{0\leqslant m\leqslant n} \frac{\mu_n^X(X_m + A)}{\log n} = -1/\log(1 - 1/\Lambda_A) \quad \text{a.s.} \tag{1.4}$$

For our first example, when $A = \{0\}$, $\Lambda_A = G(0) = 1/\gamma_d$, where γ_d is the probability of no-return to the origin, and in the case of the simple random walk we recover Theorem 13 of [3].

Here are some other examples. Set $t_y = \mathbf{P}(T_y < \infty)$, where $T_y := \inf\{s > 0 : X_s = y\}$. Let $S(0, 1) = \{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$, $B(0, 1) = \{0\} \cup S(0, 1)$, be the (Euclidean) sphere and ball in Z^d of radius 1 centered at the origin.

Theorem 1.2. *If X has finite second moments, then for any* $0 \neq y \in Z^d$

$$\lim_{n\to\infty} \sup_{x\in Z^d} \frac{\mu_n^X(x + \{0, y\})}{\log n} = -1/\log(1 - \gamma_d/(1 + t_y)) \quad \text{a.s.} \tag{1.5}$$

For the simple random walk

$$\lim_{n\to\infty} \sup_{x\in Z^d} \frac{\mu_n^X(x + S(0, 1))}{\log n} = -1/\log(1 - \gamma_d/2d(1 - \gamma_d)) \quad \text{a.s.} \tag{1.6}$$

and

$$\lim_{n\to\infty} \sup_{x\in Z^d} \frac{\mu_n^X(x+B(0,1))}{\log n} = -1/\log\left(\frac{p+\sqrt{p^2+2/d}}{2}\right) \quad \text{a.s.,} \tag{1.7}$$

where $p = 1 - 1/2d(1-\gamma_d)$.

Corollary 1.3. *If X has finite second moments, then for any fixed* $K>0$

$$\lim_{n\to\infty} \frac{\max_{x,y\in Z^d:|x-y|\leqslant K}(\mu_n^X(\{x,y\}))}{\log n} < -2/\log(1-\gamma_d) \quad \text{a.s.}$$

Since the constant for one-point set in Theorem 1.1 is $-1/\log(1-\gamma_d)$, this corollary expresses the fact that any two points with individual occupation measures up to time n, both close to the maximum, should be at a distance larger than any constant $K>0$. In particular, a neighbor of a maximally visited point is not maximally visited.

Let W_t denote Brownian motion in $\mathbb{R}^d$, $d\geqslant 3$. We denote by v_T^W its *occupation measure*:

$$v_T^W(A) = \int_0^T \mathbf{1}_A(W_t)\,\mathrm{d}t$$

for all Borel sets $A \subseteq \mathbb{R}^d$. Let $K \subseteq \mathbb{R}^d$ be a fixed compact neighborhood of the origin which is the closure of its interior and set $K(x,r) = x + rK$.

As usual, we let

$$u^0(x) = \frac{c_d}{|x|^{d-2}} \tag{1.8}$$

denote the 0-potential density for $\{W_t\}$, where $c_d = 2^{-1}\pi^{-d/2}\Gamma(\frac{d}{2}-1)$. Let Λ_K^0 denote the norm of

$$R_K f(x) = \int_K u^0(x-y) f(y)\,\mathrm{d}y$$

considered as an operator from $L^2(K, \mathrm{d}x)$ to itself. If $B(x,r)$ denotes the Euclidean ball in R^d of radius r centered at x, it is known [1] that $\Lambda^0_{B(0,1)} = 2r_d^{-2}$ where r_d is the smallest positive root of the Bessel function $J_{d/2-2}$.

With the notation of the last paragraph, the analogue of Theorem 1.1 for Brownian motion and convex K is the following. For any $S \in (0,\infty)$ and any $T \in (0,\infty)$,

$$\lim_{\varepsilon\to 0} \sup_{|x|\leqslant S} \frac{v_T(K(x,\varepsilon))}{\varepsilon^2|\log\varepsilon|} = 2\Lambda_K^0 \quad \text{a.s.} \tag{1.9}$$

and

$$\lim_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant T} \frac{v_T(K(W_t,\varepsilon))}{\varepsilon^2|\log\varepsilon|} = 2\Lambda_K^0 \quad \text{a.s.} \tag{1.10}$$

Eqs. (1.9) and (1.10) generalize statements (1.7) and (1.9) of Theorem 1.3 in [2] where instead of an arbitrary convex K only balls were considered. As the proof would be very similar we omit it.

For any $x \in R^d$ and $\varepsilon > 0$, let $e_\varepsilon(x) = x + [0, \varepsilon]^d$, the cube of edgelength ε with 'lower' corner at x. Set

$$\mathscr{L}_\varepsilon(K) = \{x \in \varepsilon Z^d \mid e_\varepsilon(x) \subseteq K\} \quad \text{and} \quad \mathscr{C}_\varepsilon(K) = \bigcup_{x \in \mathscr{L}_\varepsilon(K)} e_\varepsilon(x) \tag{1.11}$$

and assume that

$$\lim_{\varepsilon \to 0} \lambda^d(\mathscr{C}_\varepsilon(K)) = \lambda^d(K), \tag{1.12}$$

where λ^d denotes Lebesgue measure. Note that $\varepsilon^{-1}\mathscr{L}_\varepsilon(K) \subseteq Z^d$.

In view of Theorem 1.1 and (1.9)–(1.10), the next theorem gives an invariance principle. It reveals a limiting relationship between the largest eigenvalue belonging to the discretized and scaled version of the set K in the random walk case and the corresponding largest eigenvalue belonging to K in the Brownian case.

Theorem 1.4. *Assume that X_1 has $d-1$ moments and covariance matrix equal to the identity. Then*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \Lambda_{\varepsilon^{-1}\mathscr{L}_\varepsilon(K)} = \Lambda_K^0 \tag{1.13}$$

and consequently

$$-\lim_{\varepsilon \to 0} \varepsilon^2 / \log(1 - 1/\Lambda_{\varepsilon^{-1}\mathscr{L}_\varepsilon(K)}) = \Lambda_K^0. \tag{1.14}$$

Section 2 states and proves the crucial Localization Lemma 2.2. Theorem 1.1 is proven in Section 3, Theorem 1.2 and Corollary 1.3 are proven in Section 4, and Theorem 1.4 is proven in Section 5.

2. Localization for random walk occupation measures

We start by providing a convenient representation of the law of the total occupation measure $\mu_\infty^X(A)$. This representation is the counterpart of the Ciesielski–Taylor representation for the total occupation measure of spatial Brownian motion in [1, Theorem 1].

Let $(f, g)_A = \sum_{x \in A} f(x) g(x)$, and let $\delta_0(x)$ be the function on A defined for all $x \in A$ by

$$\delta_0(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Recall that $q_n(x) = \mathbf{P}(X_n = x)$.

Lemma 2.1. *Let $\{X_n\}$ be a symmetric transient random walk in Z^d, and let A be a finite set in Z^d which contains the origin. Then,*

$$\mathbf{P}(\mu_\infty^X(A)>u)=\sum_j h_j\left(\frac{\lambda_j-1}{\lambda_j}\right)^u \quad u=0,1,\dots, \tag{2.1}$$

where $\lambda_1>\lambda_2\geqslant\cdots\geqslant\lambda_{|A|}\geqslant\frac{1}{2}$ are the eigenvalues of the symmetric matrix G_A with the corresponding orthonormal eigenvectors $\phi_j(y)$, $h_j:=(1,\phi_j)_A\phi_j(0)$.

Proof of Lemma 2.1. Let $\mathscr{J}=\mu_\infty^X(A)$ and set $\bar G(x-y)=\sum_{k=1}^\infty q_k(x-y)=G(x-y)-q_0(x-y)$.

Note that for any m,

$$\begin{aligned}\mathbb{E}(\mathscr{J}^m)&=\mathbb{E}\left(\left\{\sum_{i=0}^\infty \mathbf{1}_A(X_i)\right\}^m\right)=\sum_{i_1,\dots,i_m=0}^\infty \mathbb{E}\left(\prod_{j=1}^m \mathbf{1}_A(X_{i_j})\right)\\&=\sum_{k=1}^m\sum_{\substack{c_1,\dots,c_k\in[1,m]\\c_1+\cdots+c_k=m}}\binom{m}{c_1,\dots,c_k}\sum_{A^k}\sum_{0\leqslant n_1<\cdots<n_k<\infty}\prod_{j=1}^k q_{n_j-n_{j-1}}(x_j-x_{j-1}).\end{aligned} \tag{2.2}$$

Here, k is the number of distinct indices $n_1<\cdots<n_k$ among the indices $i_1,\dots,i_m$ and c_l is the number of times that n_l appears, i.e. $c_l=|\{1\leqslant j\leqslant m\,|\,i_j=n_l\}|$. The factor $\binom{m}{c_1,\dots,c_k}$ is the number of ways to assign the value n_l to c_l of the indices $i_1,\dots,i_m$, for each $1\leqslant l\leqslant k$.

Also, we have that

$$\begin{aligned}\sum_{A^k}\sum_{0\leqslant n_1<\cdots<n_k<\infty}\prod_{j=1}^k q_{n_j-n_{j-1}}(x_j-x_{j-1})&=\sum_{A^k}G(x_1)\prod_{j=2}^k\bar G(x_j-x_{j-1})\\&=(1,\bar G_A^{k-1}G_A\delta_0)_A.\end{aligned} \tag{2.3}$$

Hence (we justify the computations shortly)

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\zeta\mathscr{J}})&=1+\sum_{m=1}^\infty\frac{\zeta^m}{m!}\sum_{k=1}^m\sum_{\substack{c_1,\dots,c_k\in[1,m]\\c_1+\cdots+c_k=m}}\binom{m}{c_1,\dots,c_k}(1,\bar G_A^{k-1}G_A\delta_0)_A\\&=1+\sum_{k=1}^\infty\sum_{m=k}^\infty\sum_{\substack{c_1,\dots,c_k\in[1,m]\\c_1+\cdots+c_k=m}}\prod_{j=1}^k\frac{\zeta^{c_j}}{c_j!}(1,\bar G_A^{k-1}G_A\delta_0)_A\\&=1+\sum_{k=1}^\infty(\mathrm{e}^\zeta-1)^k(1,\bar G_A^{k-1}G_A\delta_0)_A.\end{aligned} \tag{2.4}$$

G_A is a symmetric matrix. Let $\psi(p)$ denote the characteristic function of X_1. Then $\psi(p)$ is real and $|\psi(p)|\leqslant 1$. Thus $0\leqslant 1-\psi(p)\leqslant 2$, or equivalently $\frac{1}{1-\psi(p)}\geqslant\frac{1}{2}$. Hence, using the Fourier transform representation $G(x-y)=\int e^{i((x-y)\cdot p)}(1-\psi(p))^{-1}\,dp$ we can see that $\sum_{x,y\in A} G_A(x,y)a_x a_y \geqslant \frac{1}{2}\sum_{x\in A} a_x^2$ for any $\{a_x \in R^1; x\in A\}$. By the standard theory for symmetric matrices, G_A has all eigenvalues $\geqslant \frac{1}{2}$, and the corresponding eigenvectors of G_A, denoted $\{\phi_j\}$ form an orthonormal basis of $L^2(A)$ (see [7, Theorems VI.15 and VI.16]). Moreover, since the entries of G_A are strictly positive, by the Perron–Frobenius Theorem, see [8, Theorem XIII.43], the eigenspace corresponding to $\Lambda_A=\lambda_1$ is one dimensional, and we may and shall choose ϕ_1 such that $\phi_1(y)>0$ for all $y\in A$.

Thus we can write (2.4) as

$$\begin{aligned}\mathbb{E}(e^{\zeta\mathscr{I}}) &= 1+\sum_{k=1}^{\infty}(e^{\zeta}-1)^k(1,\bar{G}_A^{k-1}G_A\delta_0)_A\\ &= 1+\sum_{j=1}^{|A|}(1,\phi_j)_A(\phi_j,\delta_0)_A\sum_{k=1}^{\infty}(e^{\zeta}-1)^k(\lambda_j-1)^{k-1}\lambda_j\\ &= 1+\sum_{j=1}^{|A|}h_j(e^{\zeta}-1)\lambda_j\sum_{k=1}^{\infty}(e^{\zeta}-1)^{k-1}(\lambda_j-1)^{k-1}, \qquad (2.5)\end{aligned}$$

where $h_j=(1,\phi_j)_A(\phi_j,\delta_0)_A$. It is now easy to see that we can justify the derivation of (2.4) and (2.5) if

$$|(e^{\zeta}-1)(\lambda_j-1)|<1, \quad \forall j. \qquad (2.6)$$

In that case we can write (2.5) as

$$\mathbb{E}(e^{\zeta\mathscr{I}})=1+\sum_j h_j\frac{(e^{\zeta}-1)\lambda_j}{1-(e^{\zeta}-1)(\lambda_j-1)}. \qquad (2.7)$$

Since $\sum_{j=1}^{|A|}h_j=\sum_{j=1}^{|A|}(1,\phi_j)_A(\phi_j,\delta_0)_A=(1,\delta_0)_A=1$ we have that

$$\mathbb{E}(e^{\zeta\mathscr{I}})=\sum_j h_j\frac{e^{\zeta}}{1-(e^{\zeta}-1)(\lambda_j-1)}. \qquad (2.8)$$

Let $f_j=1-1/\lambda_j=(\lambda_j-1)/\lambda_j$. A straightforward calculation shows that

$$\frac{e^{\zeta}(1-f_j)}{1-e^{\gamma}f_j}=\frac{e^{\zeta}}{1-(e^{\zeta}-1)(\lambda_j-1)}, \qquad (2.9)$$

so that

$$\mathbb{E}(e^{\zeta\mathscr{I}})=\sum_j h_j\,\frac{e^{\zeta}(1-f_j)}{1-e^{\zeta}f_j}. \qquad (2.10)$$

Note that since all $\lambda_j\geqslant\frac{1}{2}$ we have $|f_j|\leqslant 1$. We can always choose ζ so that addition to (2.6) we also have

$$|e^{\zeta}|<1. \qquad (2.11)$$

Then we can write

$$\frac{e^{\zeta}(1-f_j)}{1-e^{\zeta}f_j} = e^{\zeta}(1-f_j)\sum_{k=0}^{\infty} e^{k\zeta}f_j^k. \tag{2.12}$$

Hence

$$\sum_{k=1}^{\infty} e^{k\zeta}\mathbf{P}(\mathscr{J}=k) = \mathbb{E}(e^{\zeta\mathscr{J}}) = \sum_j h_j(1-f_j)\sum_{k=1}^{\infty} e^{k\zeta}f_j^{k-1}. \tag{2.13}$$

We can choose $\zeta_0<0$ so that (2.6) and (2.11) hold. Furthermore, both sides of (2.13) are analytic functions of ζ in some neighborhood of $\zeta_0+\mathrm{i}R^1$ and agree for $\zeta_0+\mathrm{i}y$ when y is small. This is enough to allow us to conclude that

$$\mathbf{P}(\mathscr{J}=k) = \sum_j h_j(1-f_j)f_j^{k-1}, \quad k=1,2,\dots\ . \tag{2.14}$$

Hence

$$\mathbf{P}(\mathscr{J}>u) = \sum_j h_j f_j^u, \quad u=0,1,\dots\ . \tag{2.15}$$

This completes the proof of (2.1). $\square$

With the aid of (2.1) we next provide a localization result for the occupation measure of $\{X_n\}$.

Lemma 2.2 (*The localization lemma*). *Let $\{X_n\}$ be a symmetric transient random walk in Z^d with finite second moments, and let A be a finite set in Z^d. Set $\theta^* = \log(\Lambda_A/(\Lambda_A-1))$. Then for some $1<c_1<\infty$, $n\geqslant u^6$, and all $u>0$ sufficiently large*

$$c_1^{-1}e^{-\theta^* u} \leqslant \mathbf{P}(\mu_n^X(A)\geqslant u) \leqslant \mathbf{P}(\mu_\infty^X(A)\geqslant u) \leqslant c_1 e^{-\theta^* u}. \tag{2.16}$$

Proof of Lemma 2.2. Let $\mathscr{J}_n := \mu_n^X(A)$. Assume first that A contains the origin. The dominant terms in (2.15) correspond to the f_j's with largest absolute value. But since $\mathbf{P}(\mathscr{J}>u)\geqslant 0$ and is monotone decreasing in u, it is clear that these dominant terms must in addition be those which correspond to positive f_j's. Thus the f_j's with largest absolute value are positive, i.e. correspond to λ_j's which are greater than 1. Recall that ϕ_1 is a strictly positive function on A, hence in (2.15) we have $h_1>0$. Since $(x-1)/x = 1-1/x$ is strictly monotone increasing on $(1,\infty)$ we conclude that the dominant term in (2.15) is precisely the single term corresponding to the largest eigenvalue $\lambda_1 = \Lambda_A$. Hence

$$\mathbf{P}(\mathscr{J}>u) \sim h_1 f_1^u = h_1\left(\frac{\Lambda_A-1}{\Lambda_A}\right)^u = h_1 e^{-u\log(\Lambda_A/(\Lambda_A-1))} \tag{2.17}$$

implying that

$$\lim_{u\to\infty} \mathbf{P}(\mathscr{J} > u)\mathrm{e}^{\theta^* u} = h_1 \in (0,\infty) \tag{2.18}$$

out of which the upper bound of (2.16) immediately follows.

Turning to prove the corresponding lower bound, let $\tau_z := \inf\{s : |X_s| > z\}$, and note that

$$\mathbf{P}(\tau_z > n) \leqslant c_1 \exp(-c_2 n z^{-2}). \tag{2.19}$$

Here is a simple proof:

$$\begin{aligned}\mathbf{P}(\tau_z > n) &= \mathbf{P}(|X_k| \leqslant z; 1 \leqslant k \leqslant n)\\ &\leqslant \mathbf{P}(|X_{lz^2}| \leqslant z; 1 \leqslant l \leqslant nz^{-2})\\ &\leqslant \mathbf{P}(|X_{lz^2} - X_{(l-1)z^2}| \leqslant 2z; 1 \leqslant l \leqslant nz^{-2})\\ &\leqslant \prod_{l=1}^{[nz^{-2}]} \mathbf{P}(|X_{lz^2} - X_{(l-1)z^2}| \leqslant 2z)\\ &= (\mathbf{P}(|X_{z^2}| \leqslant 2z))^{[nz^{-2}]} \leqslant \mathrm{e}^{-c_2[nz^{-2}]}.\end{aligned} \tag{2.20}$$

Hence

$$\mathbf{P}(\mathscr{J}_n > u) \geqslant \mathbf{P}(\mathscr{J}_{\tau_z} > u) - \mathbf{P}(\tau_z > n) \geqslant \mathbf{P}(\mathscr{J}_{\tau_z} > u) - c^{-1}\exp(-cnz^{-2}). \tag{2.21}$$

As usual we use the notation $\mathbf{P}^a$ to denote probabilities of the random walk $a + X_n,\ n = 0, 1, \ldots$. We now observe that

$$\sup_{a\in A} \mathbf{P}^a(\mu_\infty^X(A) > u) \leqslant c\mathbf{P}(\mu_\infty^X(A) > u) \tag{2.22}$$

for some $c < \infty$ and all u. To see this, note that for each $a \in A$ we can find some n_a with $h_a = \mathbf{P}(X_{n_a} = a) > 0$. Then using the Markov property,

$$\mathbf{P}(\mu_\infty^X(A) > u) \geqslant \mathbf{P}(\{\mu_\infty^X(A) > u\} \circ \theta_{n_a}, X_{n_a} = a) = h_a \mathbf{P}^a(\mu_\infty^X(A) > u). \tag{2.23}$$

Then (2.22) follows with $c = \sup_{a\in A} h_a^{-1} < \infty$.

Let $\mathscr{J}'$ denote an independent copy of $\mathscr{J}$ and $T_A := \inf\{s > 0 : X_s \in A\}$. Noting (2.22), and using the strong Markov property, it is not hard to verify that

$$\mathbf{P}(\mathscr{J} > u) \leqslant \mathbf{P}(\mathscr{J}_{\tau_z} > u) + c\mathbf{P}(\mathscr{J} + \mathscr{J}' > u) \sup_{|v|>z} \mathbf{P}^v(T_A < \infty) \tag{2.24}$$

(cf. [2, (3.6) and (3.7)] where this is obtained for the Brownian motion). It follows from Theorem 10.1 of [6] that

$$G(x) \leqslant \frac{c}{|x|}, \quad |x| \neq 0. \tag{2.25}$$

Using this together with the fact that $G(X_{n\wedge T_A})$ is a martingale shows that

$$G(v) = \mathbb{E}^v(G(X_{T_A}), T_A < \infty) \geqslant \inf_{a\in A} G(a)\mathbf{P}^v(T_A < \infty). \tag{2.26}$$

Therefore

$$\sup_{|v|>z} \mathbf{P}^v(T_A<\infty)\leqslant cz^{-1}. \tag{2.27}$$

By (2.18) it follows that for some constant C independent of u, which may change from line to line,

$$\begin{aligned}
\mathbf{P}(\mathscr{J}+\mathscr{J}'>u) &= \mathbf{P}(\mathscr{J}>u)+\sum_{y=0}^{u}\mathbf{P}(\mathscr{J}'>u-y)\mathbf{P}(\mathscr{J}=y)\\
&\leqslant C\left[\exp(-u\theta^*)+\sum_{y=0}^{u}\exp(-(u-y)\theta^*)\mathbf{P}(\mathscr{J}=y)\right]\\
&\leqslant C\exp(-u\theta^*)+C\sum_{y=0}^{u}\exp(-u\theta^*)\\
&= C(1+u)\exp(-u\theta^*).
\end{aligned}\tag{2.28}$$

Hence, taking $z=u^2$ one gets from (2.21) and (2.24) that for some $c'>0$, all $n\geqslant u^6$ and u sufficiently large

$$\mathbf{P}(\mathscr{J}_n>u)\geqslant c'\mathrm{e}^{-\theta^* u} \tag{2.29}$$

as needed to complete the proof of the lemma when A contains the origin. In general we have

$$\begin{aligned}
\mathbf{P}(\mu_\infty^X(A)>u) &= \mathbf{P}(\{\mu_\infty^X(A)>u\}\circ\theta_{T_A}, T_A<\infty)\\
&= \sum_{a\in A}\mathbf{P}^a(\mu_\infty^X(A)>u)\mathbf{P}(T_A=T_a<\infty)
\end{aligned}\tag{2.30}$$

and since it is easy to see from its proof that (2.18) holds with $\mathbf{P}$ replaced by $\mathbf{P}^a$ for any $a\in A$, for some $c_1=c_1(a)$ it follows that (2.18) also holds. This completes the proof of the lemma. $\square$

Remark 2.3. If A is replaced by $z+A$ for some fixed $z\in Z^d$, note from (1.2) that as matrices, $G_{z+A}=G_A$. Hence $\Lambda_{z+A}=\Lambda_A$.

3. Proof of Theorem 1.1

Given Lemma 2.2, Theorem 1.1 follows by the methods of [3, Section 7]. We spell out the details.

We first prove the lower bound for (1.4). To this end fix $a<\theta^{*-1}=1/\log(\Lambda_A/(\Lambda_A-1))$.

Let $k(n)=(\log n)^8$ and $N_n=[n/k(n)]$, and $t_{i,n}=ik(n)$ for $i=0,\dots,N_n-1$. Writing $X_s^t=X_{s+t}-X_t$ it follows that

$$\sup_{m\in[0,n]}\mu_n^X(X_m+A)\geqslant \max_{0\leqslant i\leqslant N_n-1} Z_i^{(n)},$$

where $Z_i^{(n)} = \mu_{k(n)}^{X^{\prime t_{i,n}}}(A)$ are i.i.d. and by Lemma 2.2, for some $c>0$ and all n large enough,

$$\mathbf{P}\left(\max_{0\leqslant i\leqslant N_n-1} Z_i^{(n)} \leqslant a\log n\right) \leqslant (1-cn^{-a\theta^*})^{N_n} \leqslant \mathrm{e}^{-cn^{-a\theta^*}N_n}.$$

Since $a\theta^*<1$ this is summable, so that applying Borel–Cantelli, then taking $a\uparrow {\theta^*}^{-1}$, we see that a.s.

$$\liminf_{n\to\infty} \sup_{m\in[0,n]} \frac{\mu_n^X(X_m+A)}{\log n} \geqslant {\theta^*}^{-1}. \tag{3.1}$$

This gives the lower bound for (1.4).

For the upper bound, fix $a>{\theta^*}^{-1}$. Note that for any $m\in[0,n]$

$$\begin{aligned}\mu_n^X(X_m+A) &= \sum_{j=0}^{n} \mathbf{1}_{X_m+A}(X_j) = \sum_{j=0}^{n} \mathbf{1}_A(X_j - X_m)\\ &= \sum_{j=0}^{m-1} \mathbf{1}_A(X_j-X_m) + \sum_{j=m}^{n} \mathbf{1}_A(X_j-X_m)\\ &\overset{\text{law}}{=} \sum_{j=1}^{m} \mathbf{1}_A(X'_j) + \sum_{j=0}^{n-m} \mathbf{1}_A(X''_j)\end{aligned} \tag{3.2}$$

where $\{X'_j,\ j=0,1,\ldots\}$, $\{X''_j, j=0,1,\ldots\}$ are two independent copies of $\{X_j, j=0,1,\ldots\}$ and we have used the symmetry of X_1. Using this and (2.28),

$$\begin{aligned}&\mathbf{P}\left(\sup_{m\in[0,n]} \mu_n^X(X_m+A)\geqslant a\log n\right)\\ &\quad\leqslant \sum_{m=0}^{n} \mathbf{P}(\mu_n^X(X_m+A)\geqslant a\log n)\\ &\quad\leqslant 2n\mathbf{P}(\mathscr{J}+\mathscr{J}'\geqslant a\log n) \leqslant c(\log n)n^{-(a\theta^*-1)}.\end{aligned} \tag{3.3}$$

Thus letting $n_k = n^k$ for k sufficiently large that $k(a\theta^*-1)>2$, we see from applying Borel–Cantelli, then taking $a\downarrow {\theta^*}^{-1}$, that a.s.

$$\limsup_{n\to\infty} \sup_{m\in[0,n^k]} \frac{\mu_{n^k}^X(X_m+A)}{\log n^k} \leqslant {\theta^*}^{-1}.$$

The upper bound for (1.4) then follows by interpolation.

The lower bound for (1.3) follows immediately from (3.1). As for the upper bound in (1.3), we note that $\mu_n^X(x+A)=0$ unless $X_m\in x+A$ for some $m\in[0,n]$. Thus the only relevant x's in (1.3) are of the form $X_m - a$ for some $m\in[0,n]$ and $a\in A$. Thus

$$\sup_{x\in Z^d} \mu_n^X(x+A) = \sup_{m\in[0,n],a\in A} \mu_n^X(X_m-a+A). \tag{3.4}$$

Recalling Remark 2.3 and the fact that A is a finite set, the upper bound for (1.3) now follows as in the proof of the upper bound for (1.4).

4. Examples

Proof of (1.5): When $A=\{0,y\}$ we have

$$G_A=\begin{pmatrix} G(0) & G(y) \\ G(y) & G(0) \end{pmatrix}.$$

The eigenvalues are $G(0)+G(y), G(0)-G(y)$ so that $\Lambda_A=G(0)+G(y)=G(0)(1+t_y)=(1+t_y)/\gamma_d$, where $t_y=\mathbf{P}(T_y<\infty)$, γ_d is the probability of no-return to the origin, and we have used the fact that $G(y)=t_yG(0)$. Then $1-1/\Lambda_A=1-\gamma_d/(1+t_y)$.

We note that in the notation of Lemma 2.1, $h_1=1, h_2=0$ so that by (2.1)

$$\mathbf{P}(\mu_\infty^X(\{0,y\})>u)=(1-\gamma_d/(1+t_y))^u \quad u=1,2,\dots\,. \tag{4.1}$$

Proof of (1.6): We now consider the simple random walk, and for ease of notation consider first $d=3$. Let $A=\{e_1,e_2,e_3,-e_1,-e_2,-e_3\}=S(0,1)$, the (Euclidean) sphere in Z^3 of radius 1 centered at the origin. We have

$$G_{S(0,1)}=\begin{pmatrix} G(0) & G(e_1-e_2) & G(e_1-e_3) & G(2e_1) & G(e_1+e_2) & G(e_1+e_3) \\ G(e_2-e_1) & G(0) & G(e_2-e_3) & G(e_2+e_1) & G(2e_2) & G(e_2+e_3) \\ G(e_3-e_1) & G(e_3-e_2) & G(0) & G(e_3+e_1) & G(e_3+e_2) & G(2e_3) \\ G(2e_1) & G(e_1+e_2) & G(e_1+e_3) & G(0) & G(e_1-e_2) & G(e_1-e_3) \\ G(e_2+e_1) & G(2e_2) & G(e_2+e_3) & G(e_2-e_1) & G(0) & G(e_2-e_3) \\ G(e_3+e_1) & G(e_3+e_2) & G(2e_3) & G(e_3-e_1) & G(e_3-e_2) & G(0) \end{pmatrix}.$$

Using $G(x)=t_xG(0)$, where $t_x=\mathbf{P}(T_x<\infty)$, and symmetry which allows us to set $a=:t_{e_i\pm e_j}$ for $i\neq j$ and $b=:t_{2e_i}$ we can write

$$G_{S(0,1)}=G(0)\begin{pmatrix} 1 & a & a & b & a & a \\ a & 1 & a & a & b & a \\ a & a & 1 & a & a & b \\ b & a & a & 1 & a & a \\ a & b & a & a & 1 & a \\ a & a & b & a & a & 1 \end{pmatrix}.$$

It follows from the Perron–Frobenius Theorem that the largest eigenvalue is $\Lambda_{S(0,1)}=G(0)(1+4a+b)$ with eigenvector $(1,1,1,1,1,1)$. Also, it is easy to see by symmetry that $\gamma_3=\mathbf{P}(T_{e_i}=\infty)$. Then again by symmetry $\mathbf{P}(T_{e_1}=\infty)=\frac{4}{6}\mathbf{P}(T_{e_1-e_2}=\infty)+\frac{1}{6}\mathbf{P}(T_{2e_1}=\infty)$, i.e. $6\gamma_3=4\mathbf{P}(T_{e_1-e_2}=\infty)+\mathbf{P}(T_{2e_1}=\infty)$. Hence $\Lambda_{S(0,1)}=G(0)(1+4a+b)=G(0)6(1-\gamma_3)=6(1-\gamma_3)/\gamma_3$. For the case of general $d\geqslant 3$, $G_{S(0,1)}$ is now a $2d\times 2d$ matrix, which is $G(0)$ times a matrix in which each row has a single entry of 1, a single entry of $b=:t_{2e_i}$ and $2d-2$ entries of $a=:t_{e_i\pm e_j}$, where as before $t_x=\mathbf{P}(T_x<\infty)$. It is easy to see by symmetry that $\gamma_d=\mathbf{P}(T_{e_i}=\infty)$. Also, as before, it is easy to see by symmetry $\mathbf{P}(T_{e_1}=\infty)=\frac{(2d-2)}{2d}\mathbf{P}(T_{e_1-e_2}=\infty)+\frac{1}{2d}\mathbf{P}(T_{2e_1}=\infty)$, i.e. $2d\gamma_d=\mathbf{P}(T_{2e_1}=+\infty)+(2d-2)\mathbf{P}(T_{e_1-e_2}=$

$+\infty$). Hence $\Lambda_{S(0,1)} = G(0)(1+(2d-2)a+b) = G(0)2d(1-\gamma_d) = 2d(1-\gamma_d)/\gamma_d$ for all $d \geqslant 3$.

We note that in the notation of Lemma 2.1, $h_1 = 1, h_j = 0, \forall j \neq 1$ so that by (2.1)

$$\mathbf{P}(\mu_\infty^X(S(0,1)) > u) = (1-\gamma_d/2d(1-\gamma_d))^u, \quad u = 1,2,\ldots \;. \tag{4.2}$$

Actually, (2.1) assumes that $0 \in A$ which does not hold here, but using (2.30) and symmetry we have that $\mathbf{P}(\mu_\infty^X(S(0,1)) > u) = \mathbf{P}^{e_1}(\mu_\infty^X(S(0,1)) > u)$ and (4.2) follows.

Proof of (1.7): We again consider the simple random walk. Let now $A = \{0\} \cup S(0,1) = B(0,1)$, the (Euclidean) ball in Z^d of radius 1 centered at the origin. With $s = \mathbf{P}(T_{e_i} < \infty)$ and $\bar{s} = (s,\ldots,s) \in R^{2d}$ we have

$$G_{B(0,1)} = G(0)\begin{pmatrix} 1 & \bar{s} \\ \bar{s}^t & M \end{pmatrix}$$

with M the $2d \times 2d$ matrix in the previous example. M is a self-adjoint matrix, and as mentioned the principal eigenvector is $\bar{1} = (1,\ldots,1) \in R^{2d}$. If $u_i,\ i = 1,\ldots,2d-1$ denote the other orthonormal eigenvectors of $G(0)M$ with eigenvalue $\lambda_i < \Lambda_{S(0,1)}$, then since they are orthogonal to $\bar{1}$ it is clear that $(0,u_i),\ i = 1,\ldots,2d-1$ will give us $2d-1$ orthonormal eigenvectors of $G_{B(0,1)}$ with eigenvalues $\lambda_i < \Lambda_{S(0,1)}$. The remaining (two) orthonormal eigenvectors must be of the form $(v, w\bar{1})$ and the corresponding eigenvalues will be $G(0)$ times those of the 2×2 matrix

$$L = \begin{pmatrix} 1 & 2ds \\ s & \Lambda \end{pmatrix},$$

where we abbreviate $\Lambda = \Lambda_{S(0,1)}/G(0) = 2d(1-\gamma_d)$. The eigenvalues of L are

$$\frac{(1+\Lambda) \pm \sqrt{(1+\Lambda)^2 - 4(\Lambda - 2ds^2)}}{2}, \tag{4.3}$$

so that

$$1/\Lambda_{B(0,1)} = \frac{2}{G(0)}\,\frac{(1+\Lambda) - \sqrt{(1+\Lambda)^2 - 4(\Lambda - 2ds^2)}}{4(\Lambda - 2ds^2)}. \tag{4.4}$$

Since $s = 1-\gamma_d$ we have $\Lambda - 2ds^2 = 2d\gamma_d(1-\gamma_d) = \gamma_d\Lambda$ we have

$$1/\Lambda_{B(0,1)} = \frac{(1+\Lambda) - \sqrt{(1+\Lambda)^2 - 4\gamma_d\Lambda}}{2\Lambda}, \tag{4.5}$$

so that

$$\begin{aligned} 1 - 1/\Lambda_{B(0,1)} &= \frac{(\Lambda - 1) + \sqrt{(1+\Lambda)^2 - 4\gamma_d\Lambda}}{2\Lambda} \\ &= \frac{(1-1/\Lambda) + \sqrt{(1+1/\Lambda)^2 - 4\gamma_d/\Lambda}}{2}. \end{aligned} \tag{4.6}$$

Setting $p = 1 - 1/\Lambda$ we can write this as

$$1 - 1/\Lambda_{B(0,1)} = \frac{p + \sqrt{(2-p)^2 - 4\gamma_d/\Lambda}}{2} = \frac{p + \sqrt{p^2 + 4 - 4p - 4\gamma_d/\Lambda}}{2} = \frac{p + \sqrt{p^2 + 2/d}}{2}. \tag{4.7}$$

We note that in the notation of Lemma 2.1, $h_j = 0$ for the $2d-1$ orthonormal eigenvectors of the form $(0, u_i)$, $i = 1, \ldots, 2d-1$ above. For the principle eigenvalue we have (4.7) and for the other 'surviving' eigenvalue the corresponding expression is $\frac{p-\sqrt{p^2+2/d}}{2}$. Hence by (2.1)

$$\mathbf{P}(\mu_\infty^X(B(0,1)) > u) = h_1 \left(\frac{p + \sqrt{p^2 + 2/d}}{2} \right)^u + h_2 \left(\frac{p - \sqrt{p^2 + 2/d}}{2} \right)^u, \quad u = 1, 2, \ldots, \tag{4.8}$$

where h_1, h_2 can be calculated in a straightforward manner. We observe that since $p < \sqrt{p^2 + 2/d}$, the expression in (4.8) is not a mixture of geometric random variables.

Now we prove Corollary 1.3. For any $y \in Z^d$ we have $t_y^2 < 1 - \gamma_d$, since t_y^2 is the probability that the random walk hits y and then returns to 0 in finite time which is obviously less than the probability $1 - \gamma_d$ that the random walk returns to zero in finite time. This implies $(1 + t_y - \gamma_d)^2 < (1 + t_y)^2(1 - \gamma_d)$ which in turn, implies

$$-1/\log(1 - \gamma_d/(1 + t_y)) < -2/\log(1 - \gamma_d)$$

and taking $\sup_{|y| \leqslant K}$ we obtain the Corollary 1.3.

5. The Brownian connection

This section is devoted to the proof of Theorem 1.4.

Since R_K is a convolution operator on a bounded subset of $\mathbb{R}^d$ with locally $L^1(\mathbb{R}^d, \mathrm{d}x)$ kernel, it follows easily as in [4, Corollary 12.3] that R_K is a (symmetric) compact operator on $L^2(K, \mathrm{d}x)$. Moreover, the Fourier transform relation $\int e^{\mathrm{i}(x \cdot p)} u^0(x)\, \mathrm{d}x = c|p|^{-2} > 0$ implies that R_K is strictly positive definite. By the standard theory for symmetric compact operators, R_K has discrete spectrum (except near 0) with all eigenvalues positive, and of finite multiplicity (see [7, Theorems VI.15 and VI.16]). Moreover, if we use $(f, g)_{2,K}$ to denote the inner product in $L^2(K, \mathrm{d}x)$, we have that $(f, R_K g)_{2,K} > 0$ for any non-negative, non-zero, f, g, so by the generalized Perron–Frobenius Theorem, see [8, Theorem XIII.43], the eigenspace corresponding to the largest eigenvalue, Λ_K^0, is one dimensional.

Let $R_{K,\varepsilon}$ be the operator on $L^2(K, \mathrm{d}x)$ with kernel

$$u^0_{K,\varepsilon}(x,y) = \sum_{z \neq z' \in \mathscr{L}_\varepsilon(K)} u^0(z - z') 1_{e_\varepsilon(z)}(x) 1_{e_\varepsilon(z')}(y). \tag{5.1}$$

Since the sum is over disjoint sets, it can be checked easily that for any $1 < p < d/(d-2)$, $u^0_{K,\varepsilon}(x,y)$ is bounded in $L^p(K \times K, \mathrm{d}x\,\mathrm{d}y)$ uniformly in $\varepsilon > 0$:

$$\begin{aligned} \int_{K \times K} |u^0_{K,\varepsilon}(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y &= \sum_{z \neq z' \in \mathscr{L}_\varepsilon(K)} \int_{e_\varepsilon(z) \times e_\varepsilon(z')} |u^0(z - z')|^p \,\mathrm{d}x\,\mathrm{d}y \\ &= c\varepsilon^{2d} \sum_{z \neq z' \in \mathscr{L}_\varepsilon(K)} \frac{1}{|z - z'|^{p(d-2)}} \\ &= c\varepsilon^{2d - p(d-2)} \sum_{i \neq j \in Z^d, |i|,|j| \leqslant k/\varepsilon} \frac{1}{|i - j|^{p(d-2)}} \leqslant C. \end{aligned} \tag{5.2}$$

Also $u^0_{K,\varepsilon}(x,y) \to u^0(x - y)$ as $\varepsilon \to 0$ for all $x \neq y$. Hence, using (1.12)

$$\lim_{\varepsilon \to 0} (f, R_{K,\varepsilon} f)_{2,K} = (f, R_K f)_{2,K} \quad \forall f \in C(K). \tag{5.3}$$

By Uchiyama [9]

$$G(x) = (1 + \delta(x)) u^0(x) \quad \forall x \neq 0 \tag{5.4}$$

with $\delta(x)$ bounded and $\lim_{|x| \to \infty} \delta(x) = 0$ so that

$$\varepsilon^{2-d} G(\varepsilon^{-1} x) = (1 + \delta(\varepsilon^{-1} x)) u^0(x) \quad \forall x \in \mathscr{L}_\varepsilon(K), \;\; x \neq 0. \tag{5.5}$$

Let $G_{K,\varepsilon}$ be the operator on $L^2(K, \mathrm{d}x)$ with kernel

$$v^0_{K,\varepsilon}(x,y) = \sum_{z \neq z' \in \mathscr{L}_\varepsilon(K)} \varepsilon^{2-d} G(\varepsilon^{-1}(z - z')) 1_{e_\varepsilon(z)}(x) 1_{e_\varepsilon(z')}(y). \tag{5.6}$$

Using (5.5), the same argument leading to (5.3) shows that

$$\lim_{\varepsilon \to 0} (f, G_{K,\varepsilon} f)_{2,K} = (f, R_K f)_{2,K} \quad \forall f \in C(K). \tag{5.7}$$

Furthermore, since $G(0) < \infty$, if we let $\widetilde{G}_{K,\varepsilon}$ be the operator on $L^2(K, \mathrm{d}x)$ with kernel

$$w^0_{K,\varepsilon}(x,y) = \sum_{z, z' \in \mathscr{L}_\varepsilon(K)} \varepsilon^{2-d} G(\varepsilon^{-1}(z - z')) 1_{e_\varepsilon(z)}(x) 1_{e_\varepsilon(z')}(y) \tag{5.8}$$

it follows from (5.7) that

$$\lim_{\varepsilon \to 0} (f, \widetilde{G}_{K,\varepsilon} f)_{2,K} = (f, R_K f)_{2,K} \quad \forall f \in C(K). \tag{5.9}$$

It now follows from [5, Theorem VIII.3.6,] that, if $\Lambda(\widetilde{G}_{K,\varepsilon})$ denotes the largest eigenvalue of the operator $\widetilde{G}_{K,\varepsilon}$

$$\lim_{\varepsilon \to 0} \Lambda(\widetilde{G}_{K,\varepsilon}) = \Lambda(R_K) = \Lambda^0_K. \tag{5.10}$$

If f is any eigenvector for $\widetilde{G}_{K,\varepsilon}$ with eigenvalue $\zeta > 0$, it is clear that f is in the image of $\widetilde{G}_{K,\varepsilon}$ so that we can write

$$f(x) = \sum_{z \in \mathscr{L}_\varepsilon(K)} 1_{e_\varepsilon(z)}(x) f(z) \tag{5.11}$$

and the eigenvalue equation $\widetilde{G}_{K,\varepsilon} f = \zeta f$ becomes

$$1_{e_\varepsilon(z)}(x) \sum_{z' \in \mathscr{L}_\varepsilon(K)} \int_K w^0_{K,\varepsilon}(x,y) 1_{e_\varepsilon(z')}(y) f(z')\,\mathrm{d}y = \zeta f(z) \quad \forall z \in \mathscr{L}_\varepsilon(K). \tag{5.12}$$

Noting that the dy integration picks up a factor ε^d, this implies that

$$\sum_{z' \in \mathscr{L}_\varepsilon(K)} \varepsilon^2 G(\varepsilon^{-1}(z - z')) f(z') = \zeta f(z) \quad \forall z \in \mathscr{L}_\varepsilon(K). \tag{5.13}$$

Hence $\Lambda_{\varepsilon^{-1}\mathscr{L}_\varepsilon(K)} = \varepsilon^{-2} \Lambda(\widetilde{G}_{K,\varepsilon})$. Together with (5.10) this completes the proof of Theorem 1.4.

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