GAUSSIAN PROCESSES AND THE LOCAL TIMES OF SYMMETRIC LÉVY PROCESSES

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Abstract. We give a relatively simple proof of the necessary and sufficient condition for the joint continuity of the local times of symmetric Lévy processes. This result was obtained in 1988 by M. Barlow and J. Hawkes without requiring that the Lévy processes be symmetric. In 1992 the authors used a very different approach to obtain necessary and sufficient condition for the joint continuity of the local times of strongly symmetric Markov processes, which includes symmetric Lévy processes. Both the 1988 proof and the 1992 proof are long and difficult. In this paper the 1992 proof is significantly simplified. This is accomplished by using two recent isomorphism theorems, which relate the local times of strongly symmetric Markov processes to certain Gaussian processes, one due to N. Eisenbaum alone and the other to N. Eisenbaum, H. Kaspi, M.B. Marcus, J. Rosen and Z. Shi. Simple proofs of these isomorphism theorems are given in this paper.

1. Introduction

We give a relatively simple proof of the necessary and sufficient condition for the joint continuity of the local times of symmetric Lévy processes. Let $X=\{X(t),t\in R^+\}$ be a symmetric Lévy process with values in R and characteristic function

$$Ee^{i\xi X(t)}=e^{-t\psi(\xi)}. \tag{1.1}$$

Let L^x_t denote the local time of X at $x\in R$. Heuristically, L^x_t is the amount of time that the process spends at x, up to time t. A necessary and sufficient condition for the existence of the local time of X is that

$$\int_0^\infty \frac{1}{1+\psi(\lambda)}\,d\lambda<\infty. \tag{1.2}$$

When (1.2) holds we define

$$L^x_t=\lim_{\epsilon\to 0}\int_0^t f_{\epsilon,x}(X_s)\,ds \tag{1.3}$$

where $f_{\epsilon,x}$ is an approximate delta–function at x. Specifically, we assume that $f_{\epsilon,x}$ is supported in $[x-\epsilon,x+\epsilon]$ and $\int f_{\epsilon,x}(y)\,dy=1$. Convergence in (1.3) is locally uniform in t almost surely, see Theorem 2.

When (1.2) is satisfied the process X has a transition probability density which we denote by $p_t(x,y)$. The α-potential density of X is defined as

$$\begin{aligned} u^\alpha(x,y) &= \frac{1}{\pi}\int_0^\infty p_t(x,y)e^{-\alpha t}\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\cos\lambda(x-y)}{\alpha+\psi(\lambda)}\,d\lambda. \end{aligned} \tag{1.4}$$

Since X is a symmetric Lévy process $p_t(x,y)$ and $u^\alpha(x,y)$ are actually functions of $|x-y|$. We occasionally use the notation $p_t(x-y) \stackrel{def}{=} p_t(x,y)$ and similarly for u^α. Note that $u^\alpha(\cdot)$ is the Fourier transform of an L^1 function. Consequently it is uniformly continuous on R.

The transition probability $p_t(x,y)$ is positive definite. This is a simple consequence of the Chapman–Kolmogorov equation and uses the fact that $p_t(x,y)$ is symmetric. Therefore, u^α is positive definite and hence is the covariance of a stationary Gaussian process. (It is also easy to show that u^α is positive definite directly from (1.4)).

Let $G = \{G(x), x \in R\}$ be a mean zero Gaussian process with

$$EG(x)G(y) = u^1(x,y). \tag{1.5}$$

The next theorem relates the continuity of the local times of X with that of G.

Theorem 1. *(Barlow–Hawkes, Barlow (1988)) Let $L = \{L_t^x, (x,t) \in R \times R^+\}$ be the local time process of a symmetric Lévy process X with 1–potential density $u^1(x,y)$. Then L is continuous almost surely if and only if the mean zero stationary Gaussian process $\{G(x), x \in R\}$, with covariance $u^1(x,y)$, is continuous almost surely.*

In the course of proving Theorem 1 we also show that if L is not continuous almost surely then for all $t > 0$ and x_0 in R, L_t^x is unbounded in all neighborhoods of x_0, P^{x_0} almost surely.

For the historical background of Theorem 1 see the end of Section 1, Marcus and Rosen (1992).

In Barlow (1988) necessary and sufficient conditions are given for the continuity of the local time process for all Lévy processes, not only for symmetric Lévy processes. For processes that are not symmetric these conditions can not be described in terms of Gaussian processes since u^1 in (1.5), as the covariance of a Gaussian process, must be symmetric.

The results in Barlow (1988) are not expressed the way they are in Theorem 1. Theorem 1 is the way Theorem 1, in our paper Marcus and Rosen (1992), is stated. The contribution of the latter theorem is that it holds for Markov processes with symmetric potential densities, not just for symmetric Lévy processes.

There is another important difference in the work in Barlow (1988) and the work in Marcus and Rosen (1992). In Barlow (1988) concrete conditions for continuity are obtained which imply Theorem 1 as stated. In Marcus and Rosen (1992) the comparison between local times of Lévy processes and Gaussian processes is obtained abstractly, without obtaining any conditions to verify when either class of processes is continuous. However, since necessary and sufficient conditions for the continuity of Gaussian processes are known, we have them also for the local time processes.

Let

$$\sigma(u) = \int_0^\infty \frac{\sin^2 \lambda u}{1+\psi(\lambda)}\, d\lambda < \infty. \tag{1.6}$$

and denote by $\overline{\sigma(u)}$ the non–decreasing rearrangement of $\sigma(u)$ for $u \in [0,1]$. (I.e. $\overline{\sigma(u)}$, $u \in [0,1]$, is a non–decreasing function satisfying $\{u : \overline{\sigma(u)} \le x, u \in [0,1]\} = \{u : \sigma(u) \le x, u \in [0,1]\}$.)

Corollary 1. *Let $L = \{L_t^x, (x,t) \in R \times R^+\}$ be the local time process of a symmetric Lévy process X with characteristic function given by (1.1) and let σ and $\bar{\sigma}$ be as defined in (1.6). Then L is continuous almost surely if and only if*

$$\int_0^{1/2} \frac{\overline{\sigma(u)}}{u(\log 1/u)^{1/2}}\, du < \infty. \tag{1.7}$$

In particular (1.7) holds when

$$\int_2^{\infty} \frac{\left(\int_s^\infty \frac{1}{1+\psi(\lambda)}\, d\lambda\right)^{1/2}}{s(\log 1/s)^{1/2}}\, ds < \infty \tag{1.8}$$

and (1.7) and (1.8) are equivalent when $\psi'(\lambda) \geq 0$.

The proofs of Theorem 1 in Barlow (1988) and Marcus and Rosen (1992) are long and difficult. So much so that in his recent book on Lévy processes, Bertoin (1996), Bertoin only gives the proof of sufficiency. The proof of necessity in Barlow (1988) is very technical. The proofs in Marcus and Rosen (1992) depend on an isomorphism theorem of Dynkin. The form of this isomorphism makes it difficult to apply.

We have devoted a lot of effort over the past ten years trying to simplify the proof of Theorem 1. In this paper we present such a proof. Using a new Dynkin type isomorphism theorem, obtained recently in Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999), which has a relatively short proof, we greatly simplify the proof of necessity. We also use an earlier isomorphism theorem of N. Eisenbaum, Eisenbaum (19??) to significantly shorten the proof of sufficiency given in Marcus and Rosen (1992). Furthermore, using ideas developed in Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999), we give a simple proof of Eisenbaum's isomorphism theorem. Her original proof followed the more complicated line of the proof of Dynkin's theorem given in Marcus and Rosen (1992).

Another factor that enables us to simplify the presentation in this paper is that we restrict our attention to Lévy processes. Actually, the same proofs given here extend to prove Theorem 1, Marcus and Rosen (1992), in the generality in which it is given in Marcus and Rosen (1992).

The new isomorphism theorem of Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999) has other applications to Lévy process. In Bass, Eisenbaum and Shi (1999) and Marcus (1999) it is used to show that the most visited site of a large class of Lévy processes is transient.

Section 2 provides background material on local times. In Section 3 we state the two isomorphism theorems that are at the heart of this work. The necessity part of Theorem 1 and Corollary 1 are proved in Section 4. In Section 5 the sufficiency part of Theorem 1 is given. Section 6 presents Kac's formula in a form which is convenient for the proofs of the isomorphism theorems. In Section 7 we give new and simple proofs of the isomorphism theorems subject only to a lemma which is proved in Section 8. We have tried to make this paper accessible to readers whose primary interests are in Markov processes or Gaussian processes and consequently may have included more details than some specialists might think necessary. We request your indulgence.

We are grateful to Jean Bertoin for helpful discussions.

2. Local times of Lévy processes

The material in this section is provided for background. It is fairly standard, see e.g. Bertoin (1996), V.1 and Blumenthal and Getoor (1968), V.3.

A functional A_t of the Lévy process X is called a continuous additive functional if it is continuous in $t \in R^+$, $\mathcal{F}_t$ measurable and satisfies the additivity condition

$$A_{t+s} = A_t + A_s \circ \theta_t \qquad \text{for all } s, t \in R^+. \tag{2.1}$$

Let A_t be a continuous additive functional of X, with $A_0 = 0$ and let

$$S_A(\omega) = \inf\{t \mid A_t(\omega) > 0\}. \tag{2.2}$$

We call A_t a local time of X, at $y \in R$, if $P^y(S_A = 0) = 1$ and, for all $x \neq y$, $P^x(S_A = 0) = 0$.

Theorem 2. *Let X be a Lévy process as defined in (1.1) and assume that (1.2) holds. Then for each $y \in R$ we can find a local time of X at y, denoted by L^y_t, such that*

$$E^x\left(\int_0^\infty e^{-\alpha t}\, dL^y_t\right) = u^\alpha(x, y). \tag{2.3}$$

where $u^\alpha(x, y)$ is defined in (1.4).

Furthermore, there exists a sequence $\{\epsilon_n\}$ tending to zero, such that for any finite time T, which may be random

$$L^y_t = \lim_{\epsilon_n \to 0} \int_0^t f_{\epsilon_n, y}(X_s)\, ds \tag{2.4}$$

uniformly for $t \in [0, T]$.

Proof Let θ be an independent exponential time with mean $1/\alpha$, and let W_t be the Markov process obtained by killing X_t at θ. A simple calculation shows that

$$\begin{aligned} E^x\left(\left(\int_0^\infty f_{\epsilon,y}(W_s)\, ds\right)\left(\int_0^\infty f_{\epsilon',y}(W_t)\, dt\right)\right) & \\ = \int u^\alpha(x, z_1)u^\alpha(z_1, z_2) f_{\epsilon,y}(z_1) f_{\epsilon',y}(z_2)\, dz_1\, dz_2 & \\ + \int u^\alpha(x, z_1)u^\alpha(z_1, z_2) f_{\epsilon',y}(z_1) f_{\epsilon,y}(z_2)\, dz_1\, dz_2. & \end{aligned} \tag{2.5}$$

Since $u^\alpha(x, y)$ is continuous on $R \times R$ for all $\alpha > 0$, we see that $\int_0^\infty f_{\epsilon,y}(W_s)\, ds$ converges in L^2 as $\epsilon \to 0$.

Define the right continuous W-martingale

$$M^\epsilon_t = E^x\left(\int_0^\infty f_{\epsilon,y}(W_s)\, ds | \mathcal{F}'_t\right) = \int_0^t f_{\epsilon,y}(W_s)\, ds + \int u^\alpha(W_t, z) f_{\epsilon,y}(z)\, dz. \tag{2.6}$$

where $\mathcal{F}'_t = \mathcal{F}_t \vee \sigma(\theta \wedge t)$. Doob's maximal inequality shows that M^ϵ_t converges in L^2 uniformly in $t \in R^+$. Using the uniform continuity of $u^\alpha(x, y)$ we see that the last term in (2.6) also converges in L^2 uniformly in $t \in R^+$. Consequently, we can find a sequence $\epsilon_n \to 0$ such that

$$\int_0^{t\wedge\theta} f_{\epsilon_n,y}(X_s)\, ds \tag{2.7}$$

converges almost surely, uniformly in $t \in R^+$.

Since $P(\theta > v) = e^{-\alpha v} > 0$ almost surely, it follows from Fubini's theorem that the right hand side of (2.4) converges almost surely, uniformly on $[0, v]$. We define L_t^y by (2.4). It is easy to verify that L_t^y is a continuous additive functional of X with $L_0^y = 0$.

It follows from (2.5) that, $\{\int_0^\infty f_{\epsilon_n,y}(W_s)\,ds\}$ is uniformly integrable. Therefore

$$E^x(L_\theta^y) = u^\alpha(x, y). \tag{2.8}$$

Also, clearly

$$E^x(L_\theta^y) = \alpha E^x\left(\int_0^\infty e^{-\alpha t} L_t^y\,dt\right). \tag{2.9}$$

(2.3) follows from (2.8), (2.9) and integration by parts.

To complete the proof we need only show that $S = S_{L^y}$, (see (2.2)) satisfies the conditions given below (2.2). It follows immediately from (2.4) and the right continuity of X that $P^x(S = 0) = 0$ for all $x \neq y$. We now show that $P^y(S = 0) = 1$.

Suppose that $P^y(S = 0) = 0$. Since, for any $z \in R$,

$$P^z(L_t^y > 0) \le P^z(S < t) \tag{2.10}$$

we would have $\lim_{t\to 0} P^y(L_t^y > 0) = 0$. In fact since $P^x(S = 0) = 0$ for all $x \neq y$, we would actually have

$$\lim_{t\to 0} P^z(L_t^y > 0) = 0 \qquad \forall z \in R. \tag{2.11}$$

This is not possible. To see this note that it follows from the definition of S, that for any x and $t > 0$

$$P^x(S < \infty) = P^x(L_{S+t}^y > 0\,,\, S < \infty). \tag{2.12}$$

It is easy to see that S is a stopping time. Therefore, using the additivity of $L_\cdot^y$ and the Markov property, we have

$$P^x(S < \infty) = E^x(P^{X_S}(L_t^y > 0)\,,\, S < \infty). \tag{2.13}$$

Using (2.11) in (2.13) gives us $P^x(S < \infty) = 0$ for all x. That is that $L_\cdot^y \equiv 0$ almost surely, which contradicts (2.3). Thus $P^y(S = 0) > 0$. The Blumenthal $0 - 1$ law then shows that $P^y(S = 0) = 1$.

3. Isomorphism theorems for Lévy processes

In this section we present the two isomorphism theorems that play a critical role in the proof of Theorem 1. We first consider an unconditioned isomorphism theorem due to N. Eisenbaum. It is stated below Théorème 1.3, Eisenbaum (19??). (In Eisenbaum (19??) results like Theorem 3 are also obtained in more general settings.) We state it for a symmetric Lévy process X satisfying (1.2), with local time process $L = \{L_t^x, (x,t) \in R \times R^+\}$.

Let $G = \{G(x),\, x \in R\}$ be a mean–zero Gaussian process satisfying

$$EG(x)G(y) = u^1(x, y). \tag{3.1}$$

Let φ denote an exponential random variable with mean one which is independent of X. The next theorem is called unconditioned because, in contrast to Dynkin's original isomorphism theorem, it doesn't depend explicitly on X_φ.

Theorem 3. *Let X, L, φ and G be as given immediately above. For any sequence $x_j \in R$, $j = 1, \dots$ consider $\{L_\varphi^{x_j}, j = 1, \dots\}$ and $\{G(x_j), j = 1, \dots,\}$ and let $y \in R$. For all measurable functions F on R^∞*

$$E^y E_G\left(F\left(L_\varphi^{\cdot} + \frac{(G(\cdot)+s)^2}{2}\right)\right) = E\left(\left(1+\frac{G(y)}{s}\right)F\left(\frac{(G(\cdot)+s)^2}{2}\right)\right) \tag{3.2}$$

for all $s > 0$.

We defer the proof until Section 7.

The second isomorphism theorem used in this paper is a new result which is given in Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999). It is a generalization of the second Ray–Knight Theorem for Brownian motion. Let

$$u_{\{0\}}(x,y) = \phi(x) + \phi(y) - \phi(x-y) \tag{3.3}$$

where

$$\phi(x) \overset{def}{=} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\lambda x}{\psi(\lambda)}\, d\lambda \qquad x \in R. \tag{3.4}$$

(In Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999) it is shown that when X is recurrent, $u_{\{0\}}(x,y)$ is the 0-potential density of X killed at the first time it hits 0).

It is clear that when (1.2) holds, $u_{\{0\}}(x,y)$ is continuous and symmetric. Furthermore, it follows from the identity

$$\cos\lambda(x-y) - \cos\lambda x - \cos\lambda y + 1 = \mathrm{Re}\,(e^{i\lambda x}-1)(e^{-i\lambda y}-1) \tag{3.5}$$

that $u_{\{0\}}(x,y)$ is positive definite. Let $\eta = \{\eta_x,\, x \in R\}$ be a mean–zero Gaussian process with covariance

$$E_\eta(\eta_x\,\eta_y) = u_{\{0\}}(x,y) \tag{3.6}$$

where E_η is the expectation operator for η. Also, we take P_η to be the probability measure for η.

An important process in the proof of necessity in Theorem 1 is the inverse local time at 0 of X. That is

$$\tau(t) \overset{def}{=} \inf\{s : L_s^0 > t\}. \tag{3.7}$$

Theorem 4. *Assume that the Lévy process X is recurrent and let η be the mean zero Gaussian process defined in (3.6). For any $t > 0$, under the measure $P^0 \times P_\eta$,*

$$\left\{L_{\tau(t)}^x + \frac{\eta_x^2}{2},\, x \in R\right\} \overset{law}{=} \left\{\frac{\left(\eta_x + \sqrt{2t}\right)^2}{2},\, x \in R\right\}. \tag{3.8}$$

The proof of this theorem is given in Section 7.

4. Necessary condition for continuity

We give some properties of Gaussian processes which are used in the proof of the necessity part of Theorem 1.

Lemma 1. *Let G be the Gaussian process defined in (1.5) and η the Gaussian process defined in (3.6). If G is not continuous almost surely then η is unbounded almost surely, on all intervals of R.*

Proof By (3.3)

$$E(\eta_x-\eta_y)^2 = 2\phi(x-y) = E(\eta_{(x-y)}-\eta_0)^2 \tag{4.1}$$

since (3.6) implies that $\eta_0=0$. By (1.4)

$$E(G(x)-G(y))^2 = E(G(x-y)-G(0))^2 = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos\lambda(x-y)}{1+\psi(\lambda)}\,d\lambda. \tag{4.2}$$

By (3.4) and (4.1)

$$E(\eta_x-\eta_y)^2=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos\lambda(x-y)}{\psi(\lambda)}\,d\lambda. \tag{4.3}$$

Consequently

$$\left(E(G(x)-G(y))^2\right)^{1/2}\le\left(E(\eta_x-\eta_y)^2\right)^{1/2}. \tag{4.4}$$

G is a stationary Gaussian process. Therefore, by Theorem 4.9, Chapter III, Jain and Marcus (1978), if G is not continuous almost surely then it is unbounded almost surely, on all intervals of R. The conclusion about η now follows from (4.4), see (5.5), Marcus and Shepp (1972).

Lemma 2. *Let $\{G(x),x\in R\}$ be a mean zero Gaussian process on R and $T\subset R$ a finite set. Let a be the median of $\sup_{x\in T}G(x)$ and $\sigma\stackrel{def}{=}\sup_{x\in T}(EG^2(x))^{1/2}$. Then*

$$P\left(\sup_{x\in T}G(x)\ge a-\sigma s\right)\ge 1-\Phi(s) \tag{4.5}$$

and

$$P\left(\sup_{x\in T}G(x)\le a+\sigma s\right)\ge 1-\Phi(s) \tag{4.6}$$

where

$$\Phi(s)=\frac{1}{\sqrt{2\pi}}\int_s^{\infty}e^{-u^2/2}\,du. \tag{4.7}$$

Proof These statements are consequence of Borell's Lemma. For more details see (2.3) and (2.18), Marcus and Rosen (1992).

We now use the isomorphism theorem 4 to get a sufficient condition for the local time process of the Lévy process X to be unbounded in a neighborhood of a point in R. Without loss of generality we can take this point to be 0. Theorem 4, as written applies only to recurrent processes. X is recurrent if and only if $\int_0^1 1/\psi(\lambda)\,d\lambda=\infty$. This condition, which depends on ψ at zero, is completely separate from the condition for the existence of the local time of X in (1.2), which depends on ψ at infinity. To begin we consider recurrent Lévy processes.

Lemma 3. *Let X be recurrent and let $\{L_t^x,(t,x)\in R^+\times R\}$ be the local time process of X. Let $u_{\{0\}}(x,y)$ be as defined in (3.3) and let $\{\eta_x,x\in R\}$ be a real valued Gaussian process with mean zero and covariance $u_{\{0\}}(x,y)$. Suppose that there exists a countable dense subset $C\subset R$ for which*

$$\lim_{\delta\to 0}\sup_{x\in C\cap[0,\delta]}\eta_x=\infty \qquad a.s. \quad P_\eta. \tag{4.8}$$

Then

$$\lim_{\delta\to 0}\sup_{x\in C\cap[0,\delta]} L^x_t=\infty \qquad \forall\, t>0 \qquad a.s. \quad P^0. \tag{4.9}$$

Proof Fix $t,\delta>0$ and let $T\in C\cap[0,\delta]$ be a finite set. We first note that it follows from (4.5) that

$$P_\eta\left(\sup_{x\in T}\frac{(\eta_x+\sqrt{2t}\,)^2}{2}\ge\frac{(a-\sigma s+\sqrt{2t}\,)^2}{2}\right)\ge 1-\Phi(s) \tag{4.10}$$

where a is the median of $\sup_{x\in T}\eta_x$ and $\sigma\overset{def}{=}\sup_{x\in T}(E\eta_x^2)^{1/2}=\sup_{x\in T}u^{1/2}_{\{0\}}(x,x)$. $\Phi(s)$ is given in (4.7).

By Theorem 4, under the measure $P=P^0\times P_\eta$

$$\{L^x_{\tau(t)}+\frac{1}{2}\eta_x^2,\, x\in R\}\overset{law}{=}\{\frac{1}{2}\left(\eta_x+\sqrt{2t}\right)^2,\, x\in R\}. \tag{4.11}$$

Combining (4.10) and (4.11) we see that

$$P\left(\sup_{x\in T}\left(L^x_{\tau(t)}+\frac{1}{2}\eta_x^2\right)\ge\frac{(a-\sigma s+\sqrt{2t}\,)^2}{2}\right)\ge 1-\Phi(s). \tag{4.12}$$

By the triangle inequality

$$P\left(\sup_{x\in T}L^x_{\tau(t)}\ge\frac{(a-\sigma s+\sqrt{2t}\,)^2}{2}-\sup_{x\in T}\frac{\eta_x^2}{2}\right)\ge 1-\Phi(s). \tag{4.13}$$

Also, by (4.6)

$$P_\eta\left(\sup_{x\in T}\eta_x^2\le(a+\sigma s)^2\right)\ge 1-2\Phi(s). \tag{4.14}$$

Therefore

$$P^0\left(\sup_{x\in T}L^x_{\tau(t)}\ge\sqrt{2t}a-\sigma s\left(\sqrt{2t}+2a\right)+t\right)\ge 1-3\Phi(s). \tag{4.15}$$

We can take s arbitrarily large so $\Phi(s)$ is arbitrarily close to zero. We can next take δ arbitrarily small so that σ and hence σs is as small as we like, in particular so that it is less than, say $(a\wedge\sqrt{2t})/10$. Finally, we note that because of (4.8) we can take T to be a large enough set so that $\sqrt{2t}a>M$ for any number M. Thus we see that

$$\lim_{\delta\to 0}\sup_{x\in C\cap[0,\delta]} L^x_{\tau(t)}=\infty \qquad \forall\, t>0 \qquad a.s. \quad P^0. \tag{4.16}$$

$\tau(t)$ is right continuous and by the definition of local time, $\tau(0)=0$, P^0 a.s. Therefore, for any $t'>0$ and $\epsilon>0$ we can find a $0<t<t'$ so that

$$P^0(\tau(t)<t')>1-\epsilon. \tag{4.17}$$

Since the local time is increasing in t, it follows from (4.16) that

$$\lim_{\delta\to 0}\sup_{x\in C\cap[0,\delta]} L^x_t=\infty \tag{4.18}$$

on a set of P^0 measure greater than $1-\epsilon$. This gives us (4.9).

The next theorem gives the necessity part of Theorem 1.

Theorem 5. *Let X be a symmetric Lévy process with 1–potential density $u^1(x,y)$. Let $G = \{G(y), y \in R\}$ be a mean zero Gaussian process with covariance $u^1(x,y)$. If G is not continuous almost surely, then the local time of X is unbounded on all intervals of R^+.*

Proof When X is recurrent this follows immediately from Lemmas 1 and 3. Suppose that X is transient. For $s > 0$ let $\nu[s,\infty)$ denote the Lévy measure of X. It is enough to consider ν on the half line because X is symmetric. Set $\nu[s,\infty) = \nu_1[s,\infty) + \nu_2[s,\infty)$, where $\nu_1[s,\infty) = \nu[s,\infty) - \nu[1,\infty)$ for $0 < s < 1$ and $\nu_2[s,\infty) = \nu[s,\infty)$ for $1 \le s < \infty$. Let X_1 and X_2 be independent symmetric Lévy processes with Lévy measures ν_1 and ν_2. Clearly, $X \overset{law}{=} X_1 + X_2$.

Consider these processes on $[0,T]$. X_1 is recurrent and X_2 is a pure jump process with the absolute value of all its jumps greater than or equal to one. X_2 is a process of bounded variation, see e.g. Lemma 3.2.30, Stroock (1993). Hence it only has a finite number of jumps on $[0,T]$ almost surely. Conditioned on the number of jumps of X_2 being equal to k, the position of these jumps is given by the values on k independent uniform random variables on $[0,T]$. This shows that the time of the first jump of X with absolute value greater than or equal to one is greater than zero with probability one and that $X = X_1$ up to this time.

Let ψ_1 be the Lévy exponent corresponding to ν_1. Let

$$v_{\{0\}}(x,y) = \phi_1(x) + \phi_1(y) - \phi_1(x-y) \tag{4.19}$$

where

$$\phi_1(x) \overset{def}{=} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\lambda x}{\psi_1(\lambda)}\, d\lambda \qquad x \in R \tag{4.20}$$

and let $\eta_1 = \{\eta_1(x), x \in R\}$ be a mean zero Gaussian process with covariance $v_{\{0\}}(x,y)$. When G is not continuous almost surely, η_1 is unbounded almost surely on all intervals of R. This follows immediately from the proof of Lemma 1 since $\psi_1 < \psi$ so that (4.4) holds with η replaced by η_1. Given this, it follows from Lemma 3 that (4.13) holds for the local times of X_1. But then it also holds for X, since $X = X_1$ for a strictly positive amount of time, almost surely.

Proof of Corollary 1. The Gaussian process G in Theorem 1 is stationary. (1.7) is a necessary and sufficient condition for G to be continuous, see Theorem 7.6 and Corollary 6.3, Chapter IV, Jain and Marcus (1978). It is clear from (1.4) that G has spectral density $1/(1+\psi(\lambda))$. (1.8) and the statement following it are criteria for the continuity of a stationary Gaussian process in terms of its spectrum. See Theorems 3.1 and 3.2, ChapterIV, Jain and Marcus (1978).

Remark 1. *In Proposition 1.7, Barlow (1988), Barlow reduces the proof of Theorem 1 to the recurrent case. Our proof of sufficiency works for transient as well as recurrent processes so we only need to do this when considering the necessary portion of Theorem 1. This is the easier direction because (4.4) follows easily in this direction.*

Actually in Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999), Theorem 4 is given so that it holds for both transient and recurrent processes. Using this, Lemma 3, with essentially the same proof, works in both cases. When considering Lévy processes it seems simpler to consider only recurrent processes because the extension to transient processes is simple.

5. Sufficient condition for continuity

We are considering a symmetric Lévy process X for which (1.2) is satisfied. Consequently we can associate with X a local time process $L = \{L_t^x, (x,t) \in R \times R^+\}$. X also has a 1–potential density which is denoted by $u^1(x,y)$. Let φ be an exponential random variable, with mean one, that is independent of X. We begin the proof of the sufficiency part of Theorem 1 by showing that if the mean zero Gaussian process G, with covariance $u^1(x,y)$, (defined in (1.5)), is continuous almost surely, then $\{L_\varphi^x, x \in R\}$ is continuous in a very strong sense.

Lemma 4. *Let X be a symmetric Lévy process satisfying (1.2), with local time process $\{L_t^x, (x,t) \in R \times R^+\}$ and 1-potential density $u^1(x,y)$. Let φ be an exponential random variable, with mean one, that is independent of X. Let $D \subset R$ be countable dense set. When $u^1(x,y)$ is the covariance of a mean zero continuous Gaussian process*

$$\lim_{\delta\to 0} E^y \left(\sup_{\substack{|x-z|\le\delta \\ x,z\in D\cap K}} |L_\varphi^x - L_\varphi^z| \right) = 0 \tag{5.1}$$

for any compact subset K of R.

Proof Let $\|\cdot\| \stackrel{def}{=} \sup_{\substack{|x-z|\le\delta \\ x,z\in D\cap K}} |\cdot|$ and $|\!|\!|\cdot|\!|\!| \stackrel{def}{=} \sup_{x\in D\cap\{K\cup\{K+1\}\}} |\cdot|$. It follows from (3.2) with $s=1$, that

$$\begin{aligned} E^y\|L_\varphi^x - L_\varphi^z\| &\le E\|G^2(x+1) - G^2(z+1)\| \\ &\qquad + \frac{1}{2} E\left(G(y)\|G^2(x+1) - G^2(z+1)\|\right) \\ &\le \left(2\left(E|\!|\!|G^2(x)|\!|\!|^2\right)^{1/2} + \left(E|\!|\!|G^2(x)|\!|\!|^4\right)^{1/2}\right) \\ &\qquad \left(E\|G(x+1) - G(z+1)\|^2\right)^{1/2}. \end{aligned} \tag{5.2}$$

Because G is continuous on R, all moments of its sup–norm, over a compact subset of R, are finite, see e.g. Corollary 3.2, Ledoux and Talagrand (1991). Thus by the dominated convergence theorem, applied to the last line of (5.2), we obtain (5.1).

A local time process is a family of continuous additive functionals in time. When we say that a stochastic process $\hat{L} = \{\hat{L}_t^y, (t,y) \in R^+ \times R\}$ is a version of the local time of a Markov process X we mean more than the traditional statement that one stochastic process is a version of the other. Besides this we also require that the version is itself a local time for X. That is, that for each $y \in R$, $\hat{L}^y$ is a local time for X at y.

Let us be even more precise. Let $L = \{L_t^y, (t,y) \in R^+ \times R\}$ be a local time process for X. When we say that $\hat{L} = \{\hat{L}_t^y, (t,y) \in R^+ \times R\}$ is a jointly continuous version of L we mean that for all compact sets $T \subset R^+$, $\hat{L}$ is continuous on $T \times R$ almost surely with respect to P^x, for all $x \in R$, and satisfies

$$\hat{L}_t^y = L_t^y \qquad \forall\, t \in R^+ \quad \text{a.s. } P^x \tag{5.3}$$

for each $x, y \in R$.

Following convention, we often say that a Markov process has a continuous local time, when we mean that we can find a continuous version for the local time.

The next theorem gives the sufficiency part of Theorem 1.

Theorem 6. *Let X be a symmetric Lévy process satisfying (1.2), with local time process $L = \{L^y_t, (y,t) \in R \times R^+\}$ and 1-potential density $u^1(x,y)$. Let $G = \{G(y), y \in R\}$ be a mean zero Gaussian process with covariance $u^1(x,y)$. If G is continuous almost surely, there is a version of L which is jointly continuous on $R \times R^+$.*

Proof Recall that φ is an exponential random variable with mean one. Let W be the symmetric Markov process obtained by killing X at time φ and let $\mathcal{L} = \{\mathcal{L}^y_t, (t,y) \in R^+ \times R\}$ denote the local time process of W. By (2.7) and the material immediately following it we see that $\mathcal{L}^y_t = L^y_{t\wedge\varphi}$.

Let $(\Omega, \mathcal{F}_t, P^x)$ denote the probability space of W. Consider the martingale

$$A^y_t = E^x(\mathcal{L}^y_\infty \mid \mathcal{F}_t) \tag{5.4}$$

and note that

$$\mathcal{L}^y_\infty = \mathcal{L}^y_t + \mathcal{L}^y_\infty \circ \theta_t \tag{5.5}$$

where θ_t is the shift operator on $(\Omega, \mathcal{F}_t, P^x)$. Therefore, by the Markov property

$$A^y_t = \mathcal{L}^y_t + E^x(\mathcal{L}^y_\infty \circ \theta_t \mid \mathcal{F}_t) = \mathcal{L}^y_t + E^{X_t}(\mathcal{L}^y_\infty).$$

Since $\mathcal{L}^y_\infty$ is just the local time of X at y evaluated at time φ, by (2.8) we have $E^x(\mathcal{L}^y_\infty) = u^1(x,y)$. Therefore we can write (5.6) as

$$A^y_t = \mathcal{L}^y_t + u^1(X_t, y). \tag{5.6}$$

Note that A^y_t is right continuous.

Let K be a compact subset of R, D a countable dense subset of R and F a finite subset of D. We have

$$\begin{aligned} P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} \mathcal{L}^y_t - \mathcal{L}^z_t \geq 2\epsilon) &\leq P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} A^y_t - A^z_t \geq \epsilon) \\ &\quad + P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in D\cap K}} (u^1(X_t,y)) - u^1(X_t,z)) \geq \epsilon). \end{aligned} \tag{5.7}$$

Furthermore, since

$$\sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} A^y_t - A^z_t = \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} |A^y_t - A^z_t|$$

is a right continuous, non–negative submartingale, we have that for any $\epsilon > 0$

$$\begin{aligned} P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} A^y_t - A^z_t \geq \epsilon) &\leq \frac{1}{\epsilon} E^x(\sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} \mathcal{L}^y_\infty - \mathcal{L}^z_\infty) \\ &\leq \frac{1}{\epsilon} E^x(\sup_{\substack{|y-z|\leq\delta \\ y,z\in D\cap K}} \mathcal{L}^y_\infty - \mathcal{L}^z_\infty). \end{aligned} \tag{5.8}$$

Since, as mentioned above, $\mathcal{L}^y_\infty$ is just the local time of X at y evaluated at time φ, we see from (5.1) that this last term goes to zero as δ goes to zero. Consequently, for any ϵ, $\bar{\epsilon} > 0$, we can choose a $\delta > 0$ such that

$$P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} A^y_t - A^z_t \geq \epsilon) \leq \bar{\epsilon}. \tag{5.9}$$

Using this in (5.7) we see that

$$P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in F\cap K}} \mathcal{L}_t^y - \mathcal{L}_t^z \geq 2\epsilon) \leq \bar{\epsilon} + P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in D\cap K}} (u^1(X_t,y)) - u^1(X_t,z)) \geq \epsilon). \tag{5.10}$$

u^1 is uniformly continuous on R. Therefore, we can take δ small enough so that the last term in (5.10) is equal to zero. Then, taking the limit over a sequence of finite sets increasing to D, we see for any ϵ and $\bar{\epsilon} > 0$ we can find a $\delta > 0$ such that

$$P^x(\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta \\ y,z\in D\cap K}} \mathcal{L}_t^y - \mathcal{L}_t^z \geq 2\epsilon) \leq \bar{\epsilon} \tag{5.11}$$

It now follows by the Borel–Cantelli Lemma that we can find a sequence $\{\delta_i\}_{i=1}^\infty$, $\delta_i > 0$, such that $\lim_{i\to\infty} \delta_i = 0$ and

$$\sup_{t\geq 0} \sup_{\substack{|y-z|\leq\delta_i \\ y,z\in D\cap K}} \mathcal{L}_t^y - \mathcal{L}_t^z \leq \frac{1}{2^i} \tag{5.12}$$

for all $i \geq I(\omega)$, almost surely with respect to P^x.

Fix $T < \infty$. We now show that $\mathcal{L}_t^y$ is uniformly continuous on $[0,T] \times (K \cap D)$, almost surely with respect to P^x. That is, we show that for each ω in a set of measure one, with respect to P^x, we can find an $I(\omega)$ such that for $i \geq I(\omega)$

$$\sup_{\substack{|s-t|\leq\delta_i' \\ s,t\in[0,T]}} \sup_{\substack{|y-z|\leq\delta_i' \\ y,z\in D\cap K}} |\mathcal{L}_s^y - \mathcal{L}_t^z| \leq \frac{1}{2^i} \tag{5.13}$$

where $\{\delta_i'\}_{i=1}^\infty$ is a sequence of real numbers such that $\delta_i' > 0$ and $\lim_{i\to\infty} \delta_i' = 0$.

To obtain (5.13), fix ω and assume that $i \geq I(\omega)$, so that (5.12) holds. Let $\{y_1, \ldots, y_n\}$ be a finite subset of $K \cap D$ such that

$$K \subseteq \bigcup_{j=1}^n B(y_j, \delta_{i+2}) \tag{5.14}$$

where $B(y,\delta)$ is a ball of radius δ in the Euclidean metric with center y. For each y_j, $j = 1, \ldots, n$, $\mathcal{L}_t^{y_j}(\omega)$ is the local time of $W(\omega)$ at y_j. Hence it is continuous in t and consequently, uniformly continuous on $[0,T]$. Therefore, we can find a finite increasing sequence $t_1 = 0, t_2, \ldots, t_{k-1} < T, t_k \geq T$ such that $t_m - t_{m-1} = \delta_{i+2}''$ for all $m = 1, \ldots, k$ where δ_{i+2}'' is chosen so that

$$|\mathcal{L}_{t_{m+1}}^{y_j}(\omega) - \mathcal{L}_{t_{m-1}}^{y_j}(\omega)| \leq \frac{1}{2^{i+2}} \qquad \forall\, j = 1, \ldots, n; \quad \forall\, m = 1, \ldots, k-1. \tag{5.15}$$

Let $s_1, s_2 \in [0,T]$ and assume that $s_1 \leq s_2$ and that $s_2 - s_1 \leq \delta_{i+2}''$. There exists an $1 \leq m \leq k-1$ such that

$$t_{m-1} \leq s_1 \leq s_2 \leq t_{m+1}.$$

Assume also that $y, z \in K \cap D$ satisfy $|y - z| \leq \delta_{i+2}$. We can find a $y_j \in Y$ such that $y \in B(y_j, \delta_{i+2})$. If $\mathcal{L}_{s_2}^y(\omega) \geq \mathcal{L}_{s_1}^z(\omega)$ we have

$$\begin{aligned} |\mathcal{L}_{s_2}^y(\omega) - \mathcal{L}_{s_1}^z(\omega)| &\leq |\mathcal{L}_{t_{m+1}}^y(\omega) - \mathcal{L}_{t_{m-1}}^z(\omega)| \\ &\leq |\mathcal{L}_{t_{m+1}}^y(\omega) - \mathcal{L}_{t_{m+1}}^{y_j}(\omega)| + |\mathcal{L}_{t_{m+1}}^{y_j}(\omega) - \mathcal{L}_{t_{m-1}}^{y_j}(\omega)| \\ &\quad |\mathcal{L}_{t_{m-1}}^{y_j}(\omega) - \mathcal{L}_{t_{m-1}}^y(\omega)| + |\mathcal{L}_{t_{m-1}}^y(\omega) - \mathcal{L}_{t_{m-1}}^z(\omega)| \end{aligned} \tag{5.16}$$

where we use the fact that $\mathcal{L}_t^y$ is non–decreasing in t.

The second term to the right of the last inequality in (5.16) is less than or equal to $2^{-(i+2)}$ by (5.15). It follows from (5.12) that the other three terms are also less than or equal to $2^{-(i+2)}$, since $|y-y_j| \leq \delta_{i+2}$ and $|y-z| \leq \delta_{i+2}$. Taking $\delta_i' = \delta_{i+2}'' \wedge \delta_{i+2}$ we get (5.13) on the larger set $[0,T'] \times (K \cap D)$ for some $T' \geq T$. Obviously this implies (5.13) as stated in the case when $\mathcal{L}_{s_2}^y(\omega) \geq \mathcal{L}_{s_1}^z(\omega)$. A similar argument gives (5.13) when $\mathcal{L}_{s_2}^y(\omega) \leq \mathcal{L}_{s_1}^z(\omega)$. Thus (5.13) is established.

Recall that $\mathcal{L}_t^y = L_{t\wedge\varphi}^y$. Consequently L_t^y is uniformly continuous on $[0, T\wedge\varphi] \times (K \cap D)$, almost surely with respect to P^x. Therefore by Fubini's theorem we see that

$$L_t^y \text{ is uniformly continuous on } [0,T] \times (K \cap D), \qquad P^x \quad a.s. \tag{5.17}$$

In what follows we say that a function is locally uniformly continuous on a measurable set in a locally compact metric space if it is uniformly continuous on all compact subsets of the set. Let K_n be a sequence of compact subsets of R such that $R = \cup_{n=1}^\infty K_n$. Let

$$\hat{\Omega} = \{\omega \mid L_t^y(\omega) \text{is locally uniformly continuous on } R^+ \times (R \cap D)\} \tag{5.18}$$

Let $\mathcal{R}$ denote the rational numbers. Then

$$\hat{\Omega}^c = \bigcup_{\substack{s\in\mathcal{R}\\ 1\leq n\leq\infty}} \{\omega \mid L_t^y(\omega) \text{is not uniformly continuous on } [0,s] \times (K_n \cap D)\}. \tag{5.19}$$

It follows from (5.17) that $P^x(\hat{\Omega}^c) = 0$ for all $x \in R$. Consequently

$$P^x(\hat{\Omega}) = 1 \qquad \forall x \in R. \tag{5.20}$$

We now construct a stochastic process $\hat{L} = \{\hat{L}_t^y, (t,y) \in R^+ \times R\}$ which is continuous and which is a version of L. For $\omega \in \hat{\Omega}$, let $\{\hat{L}_t^y(\omega), (t,y) \in R^+ \times R\}$ be the continuous extension of $\{L_t^y(\omega), (t,y) \in R^+ \times (R \cap D)\}$ to $R^+ \times R$. For $\omega \in \hat{\Omega}^c$ set

$$\hat{L}_t^y \equiv 0 \qquad \forall\, t, y \in R^+ \times R. \tag{5.21}$$

$\{\hat{L}_t^y, (t,y) \in R^+ \times R\}$ is a well defined stochastic process which, clearly, is jointly continuous on $R^+ \times R$.

We now show that $\hat{L}$ satisfies (5.3). To begin note that we could just as well have obtained (5.17) with D replaced by $D \cup \{y\}$ and hence obtained (5.20) with D replaced by $D \cup \{y\}$ in the definition of $\hat{\Omega}$. Therefore if we take a sequence $\{y_i\}_{i=1}^\infty$ with $y_i \in D$ such that $\lim_{i\to\infty} y_i = y$ we have that

$$\lim_{i\to\infty} L_t^{y_i} = L_t^y \qquad \text{locally uniformly on } R^+ \quad \text{a.s. } P^x. \tag{5.22}$$

By the definition of $\hat{L}$ we also have

$$\lim_{i\to\infty} L_t^{y_i} = \hat{L}_t^y \qquad \text{locally uniformly on } R^+ \quad \text{a.s. } P^x. \tag{5.23}$$

This shows that

$$\hat{L}_t^y = L_t^y \qquad \forall t \quad \text{a.s. } P^x \tag{5.24}$$

which is (5.3). This completes the proof of Theorem 6.

6. Kac's formula

We give a version of Kac's formula for the moment generating function of the local time process evaluated at certain random times ξ. The formula is used in Section 7, with ξ taken to be an independent exponential T, in the proof of first isomorphism theorem and with ξ taken to be $\tau(T)$, in the proof of the second isomorphism theorem.

Lemma 5. *Let X be a Lévy process with finite α-potential density $u^\alpha(x,y)$. Let ξ be a finite random time such that V_t, the process X_t killed at ξ, is a Markov process with continuous zero–potential density $v(x,y)$. Let Σ be the matrix with elements $\Sigma_{i,j}=v(x_i,x_j)$, $i,j=1,\ldots,n$ and let $x_1=y$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j}=\lambda_i\delta_{i,j}$. For all $\lambda_1,\ldots,\lambda_n$ suficiently small we have*

$$\begin{aligned} E^y\exp\left(\sum_{i=1}^n \lambda_i L_\xi^{x_i}\right) &= \sum_{i=1}^n\sum_{k=0}^\infty \{(\Sigma\Lambda)^k\}_{1,i} \\ &= 1+\sum_{i=1}^n\{(I-\Sigma\Lambda)^{-1}\Sigma\Lambda\}_{1,i} \end{aligned}$$

Proof Let $q_t(x,dy)$ denote the transition probabilities for V and f a real valued function. We have

$$\begin{aligned} (6.1)\; E^y(\{\int_0^\xi f(X_s)\,ds\}^k)) &= E^y(\{\int_0^\infty f(V_s)\,ds\}^k)) \\ &= k!\int_{0\le s_1\le\cdots\le s_k<\infty}\int E^y(\prod_{i=1}^k f(V_{s_i}))\prod_{i=1}^k ds_i \\ &= k!\int_{0\le s_1\le\cdots\le s_k<\infty}\int\left(\int\prod_{i=1}^k f(y_i)q_{s_1}(y,dy_1)q_{s_2-s_1}(y_1,dy_2)\cdots\right. \\ &\qquad\qquad q_{s_k-s_{k-1}}(y_{k-1},dy_k))\prod_{i=1}^k ds_i \\ &= k!\int v(y,y_1)v(y_1,y_2)\cdots v(y_{k-1},y_k)\prod_{i=1}^k f(y_i)\,dm(y_i). \end{aligned}$$

For f in (6.1) take $f_\epsilon=\sum_{j=1}^n\lambda_j f_{\epsilon,x_j}$, where $f_{\epsilon,x}$ is an approximate δ function at x. By Theorem 2 we have that $\int_0^\xi f_\epsilon(X_s)\,ds\to\sum_{i=1}^n\lambda_i L_\xi^{x_i}$ a.s. The continuity of $v(x,y)$ together with (6.1) show that $\left(\int_0^\xi f_\epsilon(X_s)\,ds\right)^k$ is uniformly integrable for any k. Hence

$$\begin{aligned} & E^y\left(\left(\sum_{j=1}^n\lambda_j L_\xi^{x_j}\right)^k\right) \\ (6.2)\qquad &= k!\sum_{j_1,\ldots,j_k=1}^n v(y,x_{j_1})\lambda_{j_1}v(x_{j_1},x_{j_2})\lambda_{j_2}v(x_{j_2},x_{j_3})\cdots \\ &\qquad v(x_{j_{k-2}},x_{j_{k-1}})\lambda_{j_{k-1}}v(x_{j_{k-1}},x_{j_k})\lambda_{j_k} \end{aligned}$$

for all k. (Actually, (6.2) holds even if v is not continuous).

Let $\beta=(v(y,x_1)\lambda_1,\dots,v(y,x_n)\lambda_n)$ and $\bar{1}$ be the transpose of an n–dimensional vector with all of its elements equal to one. Note that $\sum_{j_k=1}^{n} v(x_{j_{k-1}},x_{j_k})\lambda_{j_k}$ is an $n\times 1$ matrix with entries $\{\Sigma\Lambda\bar{1}\}_{j_{k-1}}$, $j_{k-1}=1\dots,n$. Note also that $(\Sigma\Lambda)^2\bar{1}$ is an $n\times 1$ matrix and

$$\sum_{j_{k-1}=1}^{n} v(x_{j_{k-2}},x_{j_{k-1}})\lambda_{j_{k-1}}\{\Sigma\Lambda\bar{1}\}_{j_{k-1}}=\{(\Sigma\Lambda)^2\bar{1}\}_{j_{k-2}}. \tag{6.3}$$

Iterating this relationship we get

$$\begin{aligned} E^y\left(\left(\sum_{j=1}^{n}\lambda_j L_\xi^{x_j}\right)^k\right) &= k!\beta(\Sigma\Lambda)^{k-1}\bar{1} \\ &= k!\sum_{i=1}^{n}\{(\Sigma\Lambda)^k\}_{1,i} \end{aligned} \tag{6.4}$$

where we use the facts that $x_1=y$ and $\beta(\Sigma\Lambda)^{k-1}$ is an n–dimensional vector which is the same as the first row of $(\Sigma\Lambda)^k$. It follows from this that

$$\begin{aligned} E^y\exp\left(\sum_{i=1}^{n}\lambda_i L_\xi^{x_i}\right) &= \sum_{i=1}^{n}\sum_{k=0}^{\infty}\{(\Sigma\Lambda)^k\}_{1,i} \\ &= \sum_{i=1}^{n}\left(\{I\}_{1,i}+\{\sum_{k=1}^{\infty}(\Sigma\Lambda)^k\}_{1,i}\right) \\ &= 1+\sum_{i=1}^{n}\{(I-\Sigma\Lambda)^{-1}\Sigma\Lambda\}_{1,i} \end{aligned} \tag{6.5}$$

This gives us the equations in (6.1).

7. Proofs of the Isomorphism Theorems

We begin with a routine calculation which we provide for the convenience of the reader.

Lemma 6. *Let $\zeta=(\zeta_1,\dots,\zeta_n)$ be a mean zero, n-dimensional Gaussian random variable with covariance matrix Σ. Assume that Σ is invertible. Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be an n-dimensional vector and Λ an $n\times n$ diagonal matrix with λ_j as its j-th diagonal entry. Let $u=(u_1,\dots,u_n)$ be an n-dimensional vector. We can choose $\lambda_i>0, i=1\dots,n$ sufficiently small so that $(\Sigma^{-1}-\Lambda)$ is invertible and*

$$\begin{aligned} &E\exp\left(\sum_{i=1}^{n}\lambda_i(\zeta_i+u_i)^2/2\right) \\ &\qquad=\frac{1}{(\det(I-\Sigma\,\Lambda))^{1/2}}\exp\left(\frac{u\Lambda u^t}{2}+\frac{(u\Lambda\widetilde{\Sigma}\,\Lambda u^t)}{2}\right) \end{aligned} \tag{7.1}$$

where $\widetilde{\Sigma}\overset{def}{=}(\Sigma^{-1}-\Lambda)^{-1}$ and $u=(u_1,\dots,u_n)$.

Proof We write

$$E\exp\left(\sum_{i=1}^n \lambda_i(\zeta_i+u_i)^2/2\right) = \exp\left(\frac{u\Lambda u^t}{2}\right) E\exp\left(\sum_{i=1}^n \lambda_i(\zeta_i^2/2+u_i\zeta_i)\right) \tag{7.2}$$

and

$$\begin{aligned} E\exp\left(\sum_{i=1}^n \lambda_i(\zeta_i^2/2+u_i\zeta_i)\right) &= \frac{1}{(\det\,\Sigma)^{1/2}}\int \exp\left((u\cdot\Lambda\zeta)-\frac{\zeta\left(\Sigma^{-1}-\Lambda\right)\zeta^t}{2}\right)d\zeta \\ &= \frac{(\det\,\widetilde{\Sigma})^{1/2}}{(\det\,\Sigma)^{1/2}}\widetilde{E}e^{(u\cdot\Lambda\xi)} \end{aligned} \tag{7.3}$$

where ξ is an n–dimensional Gaussian random variable with mean zero and covariance matrix $\widetilde{\Sigma}$ and $\widetilde{E}$ is expectation with respect to the probability measure of ξ. Clearly

$$\widetilde{E}e^{(u\cdot\Lambda\xi)} = \exp\left(\frac{u\Lambda\widetilde{\Sigma}\,\Lambda u^t}{2}\right). \tag{7.4}$$

Putting these together gives us (7.1).

Proof of Theorem 3 To prove this theorem it suffices to show that

$$E^{x_1}E_G\exp\left(\sum_{i=1}^n\lambda_i\left(L_\varphi^{x_i}+\frac{(G(x_i)+s)^2}{2}\right)\right) = E\left(\left(1+\frac{G(x_1)}{s}\right)\exp\left(\sum_{i=1}^n\lambda_i(G(x_i)+s)^2/2\right)\right) \tag{7.5}$$

for all $x_1,\ldots,x_n$, all $s>0$ and all $\lambda_1,\ldots,\lambda_n$ sufficiently small. We write this as

$$E^{x_1}\exp\left(\sum_{i=1}^n\lambda_i L_\varphi^{x_i}\right) = \frac{E\left((1+\frac{G(x_1)}{s})\exp\left(\sum_{i=1}^n\lambda_i(G(x_i)+s)^2/2\right)\right)}{E\exp\left(\sum_{i=1}^n\lambda_i(G(x_i)+s)^2/2\right)}. \tag{7.6}$$

As in Lemma 6, we consider the matrices Σ, Λ and $\widetilde{\Sigma}=(\Sigma^{-1}-\Lambda)^{-1}$, where $\Sigma_{i,j}=u(x_i,x_j)$ and $\Lambda_{i,j}=\lambda_i\delta(i,j)$. Using Lemma 6 we note that

$$\frac{\partial}{\partial s_1}E_G\exp\left(\sum_{i=1}^n\lambda_i(G(x_i)+s_i)^2/2\right) = \lambda_1(s_1+\sum_{j=1}^n\widetilde{\Sigma}_{1,j}\lambda_j s_j)E_G\exp\left(\sum_{i=1}^n\lambda_i(G(x_i)+s_i)^2/2\right). \tag{7.7}$$

Also, clearly

$$\frac{\partial}{\partial s_1} E_G \exp\left(\sum_{i=1}^n \lambda_i (G(x_i)+s_i)^2/2\right) \tag{7.8}$$
$$= s_1\lambda_1 E\left(\left(1+\frac{G(x_1)}{s_1}\right)\exp\left(\sum_{i=1}^n \lambda_i(G(x_i)+s_i)^2/2\right)\right).$$

Thus we see that

$$\frac{E\left((1+G(x_1)]s_1)\exp\left(\sum_{i=1}^n \lambda_i(G(x_i)+s_i)^2/2\right)\right)}{E\exp\left(\sum_{i=1}^n \lambda_i(G(x_i)+s_i)^2/2\right)} = 1+\sum_{j=1}^n \widetilde{\Sigma}_{1,j}\lambda_j s_j/s_1. \tag{7.9}$$

Consequently, the right–hand side of (7.6) is equal to

$$\begin{aligned} 1+\sum_{j=1}^n \widetilde{\Sigma}_{1,j}\lambda_j &= 1+\sum_{j=1}^n \{\widetilde{\Sigma}\Lambda\}_{1,j} \\ &= 1+\sum_{j=1}^n \{(\Sigma^{-1}-\Lambda)^{-1}\Lambda\}_{1,j} \\ &= 1+\sum_{j=1}^n \{(I-\Sigma\Lambda)^{-1}\Sigma\Lambda\}_{1,j} \end{aligned} \tag{7.10}$$

By Lemma 5 applied with $\xi=\varphi$, so that $v(x,y)=u^1(x,y)$, the left–hand side of (7.6) is also equal to this last expression. Thus the theorem is proved.

We now turn our attention to Theorem 4. It is proved in Eisenbaum, Kaspi, Marcus, Rosen and Shi (1999) in a more general framework; we give a simple direct proof here. The immediate approach to the proof of (3.8), in analogy with the proof of Theorem 3, would be to apply Kac's formula to the process which is X stopped at the random time $\tau(t)$. However, this process is not a Markov process. To get a Markov processes we consider X stopped at $\tau(T)$, where $T=T(q)$ is an independent exponential random variable with mean $1/q$. To be more specific, we consider

$$Z_t = \begin{cases} X_t & \text{if } t<\tau(T) \\ \Delta & \text{otherwise} \end{cases} \tag{7.11}$$

To use Kac's formula we require that Z has a the potential density. This is given in the next lemma, which is proved in Section 8.

Lemma 7. *Z is a Markov process. When X is recurrent Z has a 0-potential density $\widetilde{u}(x,y)$, given by*

$$\widetilde{u}(x,y) = u_{\{0\}}(x,y)+1/q \tag{7.12}$$

where $u_{\{0\}}(x,y)$ is defined in (3.3).

Proof of Theorem 4 It suffices to show that

$$E^0E_\eta \exp\left(\sum_{i=1}^n \lambda_i\left(L^{x_i}_{\tau(t)}+\frac{\eta^2_{x_i}}{2}\right)\right) = E_\eta\exp\left(\sum_{i=1}^n \lambda_i(\eta_{x_i}+\sqrt{2t})^2/2\right) \tag{7.13}$$

for all $x_1, \ldots, x_n$ and all $\lambda_1, \ldots, \lambda_n$ small. We write this as

$$E^0 \exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(t)}\right) = \frac{E_\eta \exp\left(\sum_{i=1}^n \lambda_i(\eta_{x_i} + \sqrt{2t})^2/2\right)}{E_\eta \exp\left(\sum_{i=1}^n \lambda_i \eta^2_{x_i}/2\right)}. \tag{7.14}$$

We define the matrix Σ with $\Sigma_{i,j} = u_{\{0\}}(x_i, x_j)$. As in Lemma 6, we use the notation Λ and $\widetilde{\Sigma} = (\Sigma^{-1} - \Lambda)^{-1}$, where $\Lambda_{i,j} = \lambda_i \delta(i,j)$. It follows from Lemma 6 that

$$\frac{E_\eta \exp\left(\sum_{i=1}^n \lambda_i(\eta_{x_i} + \sqrt{2t})^2/2\right)}{E_\eta \exp\left(\sum_{i=1}^n \lambda_i \eta^2_{x_i}/2\right)} = \exp\left(t\mathbf{1}\Lambda\mathbf{1}^t + t\mathbf{1}\Lambda\widetilde{\Sigma}\Lambda\mathbf{1}^t\right) \tag{7.15}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$. Note that

$$\begin{aligned} \Lambda + \Lambda\widetilde{\Sigma}\Lambda &= \Lambda + \Lambda(\Sigma^{-1} - \Lambda)^{-1}\Lambda \\ &= \Lambda + \Lambda(I - \Sigma\Lambda)^{-1}\Sigma\Lambda \\ &= \Lambda(I + (I - \Sigma\Lambda)^{-1}\Sigma\Lambda) \\ &= \Lambda(I - \Sigma\Lambda)^{-1} \end{aligned} \tag{7.16}$$

Let $K = (I - \Sigma\Lambda)^{-1}$. Then for $q > \mathbf{1}\Lambda K\mathbf{1}^t$ we have that

$$\begin{aligned} \int_0^\infty qe^{-qt} \frac{E_\eta \exp\left(\sum_{i=1}^n \lambda_i(\eta_{x_i} + \sqrt{2t})^2/2\right)}{E_\eta \exp\left(\sum_{i=1}^n \lambda_i \eta^2_{x_i}/2\right)}\, dt &= \int_0^\infty qe^{-qt} e^{\mathbf{1}\Lambda K\mathbf{1}^t t}\, dt \\ &= \frac{q}{q - \mathbf{1}\Lambda K\mathbf{1}^t} \\ &= 1 + \sum_{j=1}^\infty \left(\frac{\mathbf{1}\Lambda K\mathbf{1}^t}{q}\right)^j. \end{aligned} \tag{7.17}$$

On the other hand

$$\int_0^\infty qe^{-qt} E^0\left(\exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(t)}\right)\right) dt = E^0\left(\exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(T)}\right)\right) \tag{7.18}$$

where T is an independent exponential random variable with mean $1/q$.

We now show that the right–hand sides of (7.17) and (7.18) are equal. This shows that the Laplace transforms of the two sides of equation (7.14) are equal, which completes the proof of this theorem.

We use Lemma 5 with $\xi = \tau(T)$, so that V_t is the Markov process Z defined in (7.11). Without loss of generality we can assume that $x_1 = 0$. We have

$$E^0 \exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(T)}\right) = \sum_{i=1}^n \sum_{k=0}^\infty \{(\widetilde{C}\Lambda)^k\}_{1,i} \tag{7.19}$$

where, by Lemma 7, $\widetilde{C}_{i,j} = \Sigma_{i,j} + (1/q)$ for all i, j. Thus we have $\widetilde{C} = \Sigma + (1/q)\mathbf{1}^t\mathbf{1}$.

Since $x_1 = 0$, $C_{1,j} = 0$ for all j. Consequently we have $\widetilde{C}_{1,j} = (1/q)\mathbf{1}_j$ for all j. Therefore we can write (7.19) as

$$\begin{aligned} E^0 \exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(T)}\right) &= 1 + \frac{1}{q}\mathbf{1}\Lambda \sum_{k=1}^\infty \left(\left(\Sigma + \frac{1}{q}\mathbf{1}^t\mathbf{1}\right)\Lambda\right)^{k-1} \mathbf{1}^t. \\ &= 1 + \frac{1}{q}\mathbf{1}\Lambda\left(I - \left(\Sigma + \frac{1}{q}\mathbf{1}^t\mathbf{1}\right)\Lambda\right)^{-1} \mathbf{1}^t. \end{aligned} \tag{7.20}$$

Note that

$$
(7.21)\qquad \left(I-\left(\Sigma+\frac{1}{q}\mathbf{1}^t\mathbf{1}\right)\Lambda\right)^{-1} = \left((I-\Sigma\Lambda)-\frac{1}{q}\mathbf{1}^t\mathbf{1}\Lambda\right)^{-1} = \left(I-\frac{1}{q}K\mathbf{1}^t\mathbf{1}\Lambda\right)^{-1}K.
$$

Using this in (7.20) we see that

$$
E^0\exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{\tau(T)}\right)=1+\frac{1}{q}\mathbf{1}\Lambda\sum_{j=0}^{\infty}\left(\frac{K\mathbf{1}^t\mathbf{1}\Lambda}{q}\right)^j K\mathbf{1}^t
$$

which is the same as the right hand side of (7.17).

8. Proof of Lemma 7

Let

$$
(8.1)\qquad Q_tf(x)=E^x(f(X_t);\,t<\tau(T))
$$

The Markov property for Z_t follows from the equations

$$
\begin{aligned}
(8.2)\qquad Q_{t+s}f(x) &= E^x(f(X_{t+s});\,t+s<\tau(T))\\
&= E^x(f(X_{t+s})\,E_T(t+s<\tau(T)))\\
&= E^x(f(X_{t+s})\,E_T(T>L^0_{t+s}))\\
&= E^x(f(X_{t+s})\,e^{-qL^0_{t+s}})\\
&= E^x(f(X_{t+s})\,e^{-qL^0_s\circ\theta_t}\,e^{-qL^0_t})\\
&= E^x(E^{X_t}(f(X_s)\,e^{-qL^0_s})\,e^{-qL^0_t})\\
&= E^x\left(E^{X_t}\left(f(X_s);\,s<\tau(T)\right);\,t<\tau(T)\right)\\
&= Q_tQ_sf(x)
\end{aligned}
$$

where, clearly, E_T denotes expectation with respect to T.

Using the Markov property at the stopping time $\tau(T)$ we see that for any bounded continuous function f we have

$$
\begin{aligned}
\int u^\alpha(x,y)f(y)\,dy &= E^x\left(\int_0^\infty e^{-\alpha t}f(X_t)\,dt\right)\\
(8.3)\qquad &= E^x\left(\int_0^{\tau(T)} e^{-\alpha t}f(X_t)\,dt\right)+E^x\left(\int_{\tau(T)}^\infty e^{-\alpha t}f(X_t)\,dt\right)\\
&= E^x\left(\int_0^\infty e^{-\alpha t}f(Z_t)\,dt\right)+E^x(e^{-\alpha\tau(T)})\,E^0\left(\int_0^\infty e^{-\alpha t}f(X_t)\,dt\right)\\
&= E^x\left(\int_0^\infty e^{-\alpha t}f(Z_t)\,dt\right)+E^x(e^{-\alpha\tau(T)})\int u^\alpha(0,y)f(y)\,dy.
\end{aligned}
$$

The first term on the right–hand side of (8.3) is, by definition, $\widetilde{U}^\alpha f(x)$, where $\widetilde{U}^\alpha$ is the α–potential of Z. To say that Z has an α–potential density means that we can find a function $\widetilde{u}^\alpha(x,y)$ such that $\widetilde{U}^\alpha f(x)=\int\widetilde{u}^\alpha(x,y)f(y)\,dy$. Thus we see from (8.3) that Z has the α-potential density

$$
\begin{aligned}
(8.4)\qquad \widetilde{u}^\alpha(x,y) &= u^\alpha(x,y)-E^x(e^{-\alpha\tau(T)})\,u^\alpha(0,y)\\
&= u^\alpha(x,y)-E^x(e^{-\alpha T_0})\,E^0(e^{-\alpha\tau(T)})\,u^\alpha(0,y)
\end{aligned}
$$

where the second equality comes by writing $\tau(T) = T_0 + \tau(T) \circ \theta_{T_0}$, with $T_0 = \inf\{t > 0 : X_t = 0\}$, and applying the Markov property at T_0.

We can rewrite the second line of (8.4) as

$$
\begin{aligned}
(8.5)\quad \widetilde{u}^\alpha(x,y) &= u^\alpha(x,y) - E^x(e^{-\alpha T_0})\, u^\alpha(0,y) \\
&\qquad +(1 - E^0(e^{-\alpha\tau(T)}))\, E^x(e^{-\alpha T_0})\, u^\alpha(0,y) \\
&= u^\alpha(x,y) - E^x(e^{-\alpha T_0})\, u^\alpha(0,y) \\
&\qquad +(1 - E^0(e^{-\alpha\tau(T)}))\, E^x(e^{-\alpha T_0})\, E^y(e^{-\alpha T_0})\, u^\alpha(0,0)
\end{aligned}
$$

where the last equality comes from using the identity

$$
(8.6)\qquad E^y(e^{-\alpha T_0}) = u^\alpha(y,0)/u^\alpha(0,0).
$$

(To make the proof more complete, the simple proof of this identity is given at the end of this section).

Evaluating (8.4) with $(x,y) = (0,0)$ gives us an expression for $E^0(e^{-\alpha\tau(T)}))$. Using it in (8.5) shows that

$$
(8.7)\quad \widetilde{u}^\alpha(x,y) = u^\alpha(x,y) - E^x(e^{-\alpha T_0})\, u^\alpha(0,y) + E^x(e^{-\alpha T_0})\, E^y(e^{-\alpha T_0})\, \widetilde{u}^\alpha(0,0).
$$

Using (8.6), (see also (1.4) and (3.4), we note that

$$
\begin{aligned}
(8.8)\quad &\lim_{\alpha\to 0} \left(u^\alpha(x,y) - E^x(e^{-\alpha T_0})\, u^\alpha(0,y)\right) \\
&= \lim_{\alpha\to 0} (u^\alpha(x-y) - u^\alpha(x)) - (u^\alpha(y) - u^\alpha(0))\, E^x(e^{-\alpha T_0}) \\
&= \lim_{\alpha\to 0} (u^\alpha(0) - u^\alpha(x)) + (u^\alpha(0) - u^\alpha(y))\, E^x\left(e^{-\alpha T_0}\right) - (u^\alpha(0) - u^\alpha(x-y)) \\
&= \phi(x) + \phi(y) - \phi(x-y) \\
&= u_{\{0\}}(x,y).
\end{aligned}
$$

We now prove that

$$
(8.9)\qquad \lim_{\alpha\to 0} \widetilde{u}^\alpha(0,0) = 1/q.
$$

This implies that $\lim_{\alpha\to 0} \widetilde{u}^\alpha(x,y) = \widetilde{u}(x,y)$ exists and that (7.12) holds.

To obtain(8.9) we first note that (2.3) together with the Markov property imply that

$$
\begin{aligned}
u^\alpha(x,y) &= E^x\left(\int_0^{\tau(T)} e^{-\alpha t}\, dL_t^y\right) + E^x\left(\int_{\tau(T)}^\infty e^{-\alpha t}\, dL_t^y\right) \\
&= E^x\left(\int_0^{\tau(T)} e^{-\alpha t}\, dL_t^y\right) + E^x\left(e^{-\alpha\tau(T)} E^{X_{\tau(T)}}\left(\int_0^\infty e^{-\alpha t}\, dL_t^y\right)\right) \\
&= E^x\left(\int_0^{\tau(T)} e^{-\alpha t}\, dL_t^y\right) + E^x(e^{-\alpha\tau(T)})\, u^\alpha(0,y).
\end{aligned}
$$

Comparing this with (8.4) shows that

$$
(8.10)\qquad E^x\left(\int_0^{\tau(T)} e^{-\alpha s}\, dL_s^y\right) = \widetilde{u}^\alpha(x,y).
$$

Set $x = y = 0$ in (8.10) and take the limit as α goes to zero. Since $E^0(L^0_{\tau(T)}) = E(T)$ we get (8.9). This completes the proof of lemma 7.

We now prove (8.6). Let $\mathcal{T}_n$ be the first hitting time of $[-1/n, 1/n]$ and let $f_{n^{-1},0}$ be as defined in (1.3). We have

$$\begin{aligned} \int u^{\alpha}(y,v) f_{n^{-1},0}(v)\,dv &= E^y\left(\int_0^\infty e^{-\alpha t} f_{n^{-1},0}(X_t)\,dt\right) \\ &= E^y\left(\int_{\mathcal{T}_n}^\infty e^{-\alpha t} f_{n^{-1},0}(X_t)\,dt\right) \qquad (8.11) \\ &= E^y\left(e^{-\alpha \mathcal{T}_n}\, E^{X_{\mathcal{T}_n}}\left(\int_0^\infty e^{-\alpha t} f_{n^{-1},0}(X_t)\,dt\right)\right) \\ &= E^y\left(e^{-\alpha \mathcal{T}_n} \int u^{\alpha}(X_{\mathcal{T}_n}, v) f_{n^{-1},0}(v)\,dv\right). \end{aligned}$$

(8.6) follows from the continuity of $u^{\alpha}(y,v)$ and the fact that $\mathcal{T}_n$ increases to T_0 as n goes to infinity, almost surely.

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