

## A RAY-KNIGHT THEOREM FOR SYMMETRIC MARKOV PROCESSES

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Let  $X$  be a strongly symmetric recurrent Markov process with state space  $S$  and let  $L_t^x$  denote the local time of  $X$  at  $x \in S$ . For a fixed element 0 in the state space  $S$ , let

$$\tau(t) := \inf\{s: L_s^0 > t\}.$$

The 0-potential density,  $u_{\{0\}}(x, y)$ , of the process  $X$  killed at  $T_0 = \inf\{s: X_s = 0\}$ , is symmetric and positive definite. Let  $\eta = \{\eta_x; x \in S\}$  be a mean-zero Gaussian process with covariance

$$E_\eta(\eta_x \eta_y) = u_{\{0\}}(x, y).$$

The main result of this paper is the following generalization of the classical second Ray–Knight theorem: for any  $b \in R$  and  $t > 0$ ,

$$\left\{L_{\tau(t)}^x + \frac{1}{2}(\eta_x + b)^2; x \in S\right\} = \left\{\frac{1}{2}(\eta_x + \sqrt{2t + b^2})^2; x \in S\right\} \quad \text{in law.}$$

A version of this theorem is also given when  $X$  is transient.

**1. Introduction.** The goal of this paper is to generalize the second Ray–Knight theorem to all strongly symmetric Markov processes with finite 1-potential densities. In [13], Chapter XI, Section 2, the authors use the terminology “second Ray–Knight theorem” to describe the law of the Brownian local time process  $\{L_{\tau(t)}^x; x \geq 0\}$ , where  $\tau(t) := \inf\{s: L_s^0 > t\}$  is the right-continuous inverse of the mapping  $t \mapsto L_t^0$ . In its standard formulation, this theorem says that for any fixed  $t > 0$ , the process  $\{L_{\tau(t)}^x; x \geq 0\}$ , under the measure  $P^0$ , is a Markov process in  $x \geq 0$  and identifies this process as a squared Bessel process of dimension 0 starting at  $t$ . For a general Markov process  $X$ , the methods of [6] can be used to show that the analogous local time process will not be a Markov process in the state variable  $x$  unless  $X$  has continuous paths.

There is an alternate formulation of the second Ray–Knight theorem. It states that for any  $t > 0$ , under the measure  $P^0 \times \tilde{P}^0$ ,

$$(1.1) \quad \left\{L_{\tau(t)}^x + \frac{1}{2}\tilde{B}_x^2; x \geq 0\right\} = \left\{\frac{1}{2}(\tilde{B}_x + \sqrt{2t})^2; x \geq 0\right\} \quad \text{in law,}$$

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where  $\{\tilde{B}_x/\sqrt{2}; x \geq 0\}$  is a real-valued Brownian motion, with measure  $\tilde{P}^0$ , independent of the original Brownian motion. The equivalence of this with the standard formulation mentioned above is a consequence of the additivity property of squared Bessel processes, (see, e.g., [13], Chapter XI, Theorem 1.2). (1.1) is the version of the second Ray–Knight theorem which we generalize to strongly symmetric Markov processes by replacing  $\tilde{B}_x$  by a different mean-zero Gaussian process.

Let  $X = (\Omega, \mathcal{F}_t, X_t, P^x)$ ,  $t \in R^+$ , denote a strongly symmetric standard Markov process with state space  $S$ , which is a locally compact separable metric space, and with reference measure  $m$ . The full definition of these terms is given in [9]. Strong symmetry means that for each  $\alpha > 0$  and  $x \in S$ , the potential measures

$$(1.2) \quad U^\alpha(x, A) = \int_0^\infty e^{-\alpha t} E^x(1_A(X_t)) dt$$

are absolutely continuous with respect to the reference measure  $m$ , and we can find densities  $u^\alpha(x, y)$ , for the potential measures, which are symmetric in  $x, y$ . Throughout this paper we assume that the densities  $u^\alpha(x, y)$  are finite for some, hence all  $\alpha > 0$ . This is the necessary and sufficient condition for the existence of local times; see, for example, Theorem 3.2 of [9].

Let  $L_t^x$  denote the local time of  $X$  at  $x \in S$ . Heuristically,  $L_t^x$  is the amount of time that the process spends at  $x$ , up to time  $t$ . We can define

$$(1.3) \quad L_t^x = \lim_{\varepsilon \rightarrow 0} \int_0^t f_{\varepsilon, x}(X_s) ds,$$

where  $f_{\varepsilon, x}$  is an approximate delta function at  $x$ . Specifically, we take the support of  $f_{\varepsilon, x}$  to be  $B(x, \varepsilon)$ , the ball of radius  $\varepsilon$  centered at  $x$  and  $\int f_{\varepsilon, x}(y) dm(y) = 1$ . Convergence here is locally uniform in  $t$ , almost surely. Hence  $L_t^x$  inherits from  $\int_0^t f_{\varepsilon, x}(X_s) ds$  both continuity in  $t$  and the additivity property:  $L_{t+s}^x = L_t^x + L_s^x \circ \theta_t$ . Let 0 denote a fixed element in the state space  $S$  and let

$$(1.4) \quad \tau(t) := \inf\{s: L_s^0 > t\}.$$

$\tau(t)$  is the right-continuous inverse of the mapping  $t \mapsto L_t^0$ .

Let  $Y$  denote the process  $X$  killed when it first hits 0.  $Y$  is a strongly symmetric Markov process. We use  $u_{\{0\}}^\alpha(x, y)$  to denote its  $\alpha$ -potential densities. These densities, and in particular the 0-potential density  $u_{\{0\}}(x, y)$ , are finite and positive definite. See Lemma 5.1. Let  $\eta = \{\eta_x; x \in S\}$  be a mean-zero Gaussian process with covariance

$$(1.5) \quad E_\eta(\eta_x \eta_y) = u_{\{0\}}(x, y).$$

(We define  $P_\eta$  to be the measure for  $\eta$  and denote the corresponding expectation operator by  $E_\eta$ .)

The main result of this paper is the following generalization of the classical second Ray–Knight theorem (1.1).

**THEOREM 1.1.** *Let  $X = (\Omega, \mathcal{F}_t, X_t, P^x)$  be a strongly symmetric recurrent Markov process and let  $\{L_{\tau(t)}^x; x \in S\}$  be the local time process for  $X$  as defined above.*

*For any  $b \in R$  and  $t > 0$ , under the measure  $P^0 \times P_\eta$ ,*

$$(1.6) \quad \left\{L_{\tau(t)}^x + \frac{1}{2}(\eta_x + b)^2; x \in S\right\} = \left\{\frac{1}{2}(\eta_x + \sqrt{2t + b^2})^2; x \in S\right\} \quad \text{in law.}$$

When  $X$  is a real-valued symmetric Lévy process, the Gaussian process in Theorem 1.1 has stationary increments, with  $\eta_0 = 0$ . In particular when  $X$  is a real-valued symmetric stable process of index  $\beta \in (1, 2]$ , the Gaussian process is a fractional Brownian motion of index  $\beta - 1$ . This is shown in Section 6.

Theorem 1.1 is reminiscent of Dynkin’s isomorphism theorem [3], which is used in [9], [10], [4] and [5] to obtain many interesting facts about the local time process  $\{L_t^x; x \in S\}$ , by exploiting properties of associated Gaussian processes. However, Dynkin’s theorem refers to the total local time  $\{L_\lambda^x; x \in S\}$ , for the process killed at an independent exponential time  $\lambda$ , or to  $\{L_\infty^x; x \in S\}$  for transient processes. Because of this Dynkin’s theorem lends itself primarily to the study of properties of  $\{L_t^x; x \in S\}$  which are uniform in  $t$ . Theorem 1.1 allows one to obtain results about  $\{L_t^x; x \in S\}$  which may vary with  $t$ .

Theorem 1.1 is used in [1] to show that the most visited site up to time  $t$ , of a symmetric stable process, is transient. This result is extended to a larger class of Lévy processes in [8]. In [11], Theorem 1.1 is used to give a relatively simple proof of the necessary and sufficient condition for the joint continuity of the local time of symmetric Lévy processes. (Actually, the proofs in [11] are also valid for all strongly symmetric Markov processes.)

In Theorem 1.1 the Markov process  $X$  is taken to be recurrent. If it is transient, then with probability 1,  $L_\infty^0 < \infty$ . Consequently, (1.6) can not hold for transient processes. To see this, note that it follows immediately from (1.6) that  $L_{\tau(t)}^0 = t$  for all  $t$ , which is impossible if  $t > L_\infty^0$ .

We now present a generalization of (1.6) so that it also holds when  $X$  is transient. When  $X$  is transient, its 0-potential,  $u(0, 0)$ , is finite and, under  $P^0$ ,  $L_\infty^0$  is an exponential random variable with mean  $u(0, 0)$ . When  $X$  is recurrent, its 0-potential,  $u(0, 0)$ , is infinite. In the next theorem we define  $\rho$  to be an exponential random variable with mean  $u(0, 0)$ , when  $u(0, 0) < \infty$  and set  $\rho$  to be identically equal to  $\infty$  when  $u(0, 0) = \infty$ . Let  $\tau^-(t) = \inf\{s: L_s^0 \geq t\}$ , the left-continuous inverse of  $t \mapsto L_t^0$ . Let  $\eta$  be as defined in (1.5) and take  $\eta$  and  $\rho$  to be independent.

**THEOREM 1.2.** *Let  $X = (\Omega, \mathcal{F}_t, X_t, P^x)$  be a strongly symmetric Markov process and  $h_x := P^x(T_0 < \infty)$ . For any  $b \in R^1$  and  $t \in (0, \infty]$ , under the measure  $P^0 \times P_\eta$ , we have*

$$(1.7) \quad \left\{L_{\tau^-(t \wedge L_\infty^0)}^x + \frac{1}{2}(\eta_x + h_x b)^2; x \in S\right\} \\ = \left\{\frac{1}{2}(\eta_x + h_x \sqrt{2(t \wedge \rho) + b^2})^2; x \in S\right\} \quad \text{in law.}$$

In particular, when  $X$  is transient,

$$(1.8) \quad \begin{aligned} & \left\{ L_{\tau^-(L_\infty^0)}^x + \frac{1}{2}(\eta_x + h_x b)^2; x \in S \right\} \\ &= \left\{ \frac{1}{2}(\eta_x + h_x \sqrt{2\rho + b^2})^2; x \in S \right\} \quad \text{in law.} \end{aligned}$$

An important tool used to prove these results is a new interpretation of a familiar formula due to M. Kac for the moment generating function of the total accumulated local time process. This is given in Section 3. In Section 2 we present some preliminaries concerning Gaussian processes. In Section 4 we prove an isomorphism theorem closely related to Theorems 1.1 and 1.2, which facilitates their proofs. Section 5 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 7 we use the Kac-type formula to give an easy proof of the first Ray–Knight theorem.

To apply Theorems 1.1 and 1.2 one must obtain the 0-potential density  $u_{\{0\}}(x, y)$ , of the process  $X$  killed at  $T_0 = \inf\{s: X_s = 0\}$ , in order to identify the Gaussian process  $\eta$ , defined in (1.5). In Section 6 we do this when  $X$  is a real-valued symmetric Lévy process. It turns out that when  $X$  is recurrent,  $\eta$  has stationary increments. In Remark 6.1, we show that when  $X$  is a symmetric stable process with index  $\beta \in (1, 2]$ ,  $\eta$  is a fractional Brownian motion with index  $\beta - 1$ .

**2. Preliminaries on Gaussian processes.** In Corollary 2.1 and Lemma 2.2 we establish some equivalence relationships between the sums of squares of two independent Gaussian processes which play an important role in the proof of Theorems 1.1 and 1.2. They are consequences of the following routine calculation which we provide for the convenience of the reader.

LEMMA 2.1. *Let  $\zeta = (\zeta_1, \dots, \zeta_n)$  be a mean zero,  $n$ -dimensional Gaussian random variable with covariance matrix  $\Sigma$ . Assume that  $\Sigma$  is invertible. Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be an  $n$ -dimensional vector and  $\Lambda$  an  $n \times n$  diagonal matrix with  $\lambda_j$  as its  $j$ th diagonal entry. Let  $u = (u_1, \dots, u_n)$  be an  $n$ -dimensional vector. We can choose  $\lambda_i > 0$ ,  $i = 1, \dots, n$  sufficiently small so that  $(\Sigma^{-1} - \Lambda)$  is invertible and*

$$(2.1) \quad \begin{aligned} & E \exp \left( \sum_{i=1}^n \lambda_i (\zeta_i + u_i)^2 / 2 \right) \\ &= \frac{1}{(\det(I - \Sigma\Lambda))^{1/2}} \exp \left( \frac{(\gamma \cdot u)}{2} + \frac{(\gamma \tilde{\Sigma} \gamma^t)}{2} \right), \end{aligned}$$

where  $\tilde{\Sigma} := (\Sigma^{-1} - \Lambda)^{-1}$  and  $\gamma = (\lambda_1 u_1, \dots, \lambda_n u_n)$ .

PROOF. We write

$$\begin{aligned}
 (2.2) \quad & E \exp \left( \sum_{i=1}^n \lambda_i (\zeta_i + u_i)^2 / 2 \right) \\
 &= \exp \left( \frac{(\gamma \cdot u)}{2} \right) E \exp \left( \sum_{i=1}^n \lambda_i (\zeta_i^2 / 2 + u_i \zeta_i) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 (2.3) \quad & E \exp \left( \sum_{i=1}^n \lambda_i (\zeta_i^2 / 2 + u_i \zeta_i) \right) \\
 &= \frac{1}{(\det \Sigma)^{1/2}} \int \exp \left( (\gamma \cdot \zeta) - \frac{\zeta (\Sigma^{-1} - \Lambda) \zeta^t}{2} \right) d\zeta \\
 &= \frac{(\det \tilde{\Sigma})^{1/2}}{(\det \Sigma)^{1/2}} \tilde{E} e^{(\gamma \cdot \xi)},
 \end{aligned}$$

where  $\xi$  is an  $n$ -dimensional Gaussian random variable with mean zero and covariance matrix  $\tilde{\Sigma}$  and  $\tilde{E}$  is expectation with respect to the probability measure of  $\xi$ . Clearly,

$$(2.4) \quad \tilde{E} e^{(\gamma \cdot \xi)} = \exp \left( \frac{(\gamma \tilde{\Sigma} \gamma^t)}{2} \right).$$

Putting these together gives us (2.1).

We have the following immediate corollary of Lemma 2.1.

COROLLARY 2.1. *Let  $\eta = \{\eta_x; x \in S\}$  be a mean-zero Gaussian process and  $f_x$  a real-valued function on  $S$ . It follows from Lemma 2.1 that for  $a^2 + b^2 = c^2 + d^2$ ,*

$$\begin{aligned}
 (2.5) \quad & \{(\eta_x + f_x a)^2 + (\tilde{\eta}_x + f_x b)^2; x \in S\} \\
 &= \{(\eta_x + f_x c)^2 + (\tilde{\eta}_x + f_x d)^2; x \in S\} \quad \text{in law,}
 \end{aligned}$$

where  $\tilde{\eta}$  is an independent copy of  $\eta$ .

Let  $G = \{G_x; x \in S\}$  be a mean-zero Gaussian process with covariance  $u(x, y)$ . We use the standard decomposition to write  $G_x = \rho_x + h_x G_0$ , where  $h_x = u(x, 0)/u(0, 0)$  and  $\rho_x = G_x - h_x G_0$ . Thus  $\rho_x$  and  $h_x G_0$  are independent Gaussian processes.

LEMMA 2.2. *Let  $\tilde{G}$  be an independent copy of  $G$ . Then*

$$(2.6) \quad \left\{ \frac{G_x^2}{2} + \frac{\tilde{G}_x^2}{2}; x \in S \right\} = \left\{ \frac{\rho_x^2}{2} + \frac{(\tilde{\rho}_x + h_x \sqrt{2T})^2}{2}; x \in S \right\} \quad \text{in law,}$$

where  $T$  is an exponential random variable with mean  $EG_0^2$ , independent of  $G$  and  $\tilde{G}$ .

PROOF. Let  $E_\rho$  denote expectation with respect to  $\{\rho_x; x \in S\}$  and  $E_{G_0}$  denote expectation with respect to  $G_0$ . Clearly,  $E_G = E_\rho E_{G_0}$ . Let  $\gamma = (\gamma_1, \dots, \gamma_m)$  and  $\tau = (\tau_1, \dots, \tau_m)$  be two vectors. We write  $\gamma \cdot \tau^2 = \sum_{i=1}^m \gamma_i \tau_i^2$ . Let  $\lambda = (\lambda_0, \dots, \lambda_n)$  and  $\tilde{\lambda} = (\lambda_1, \dots, \lambda_n)$ . Let  $G = (G_{x_0}, \dots, G_{x_n})$  and let  $h = (h_{x_0}, \dots, h_{x_n})$ , where  $x_0 = 0$ . Note that for all  $\lambda_0, \dots, \lambda_n$  sufficiently small we have

$$(2.7) \quad E_G e^{\lambda \cdot G^2/2} E_{\tilde{G}} e^{\tilde{\lambda} \cdot \tilde{G}^2/2} = E_{G_0} E_{\tilde{G}_0} \left( e^{\lambda_0 h_0^2 (G_0^2 + \tilde{G}_0^2)/2} (E_\rho e^{\tilde{\lambda} \cdot (\rho + h G_0)^2/2})^2 \right),$$

where  $\rho = (\rho_{x_1}, \dots, \rho_{x_n})$ . Also, by Corollary 2.1,

$$(2.8) \quad (E_\rho e^{\tilde{\lambda} \cdot (\rho + h G_0)^2/2})^2 = E_\rho e^{\tilde{\lambda} \cdot \rho^2/2} E_\rho e^{\tilde{\lambda} \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2}.$$

Combining these we see that the left-hand side of (2.7) equals

$$(2.9) \quad E_\rho e^{\tilde{\lambda} \cdot \rho^2/2} E_{G_0} E_{\tilde{G}_0} E_\rho e^{\lambda \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2}.$$

Last, set  $EG_0^2 = 1/q$ , and note that

$$(2.10) \quad \begin{aligned} & E_{G_0} E_{\tilde{G}_0} E_\rho e^{\lambda \cdot (\rho + h \sqrt{G_0^2 + \tilde{G}_0^2})^2/2} \\ &= \frac{q}{\pi} \int_0^\infty \int_0^\infty E_\rho e^{\lambda \cdot (\rho + h \sqrt{2(s+t)})^2/2} e^{-(s+t)q} \frac{1}{\sqrt{st}} ds dt \\ &= E e^{\lambda \cdot (\rho + h \sqrt{2T})^2/2}. \end{aligned}$$

Substituting this in (2.9) completes the proof of this lemma.  $\square$

**3. A Kac-type formula.** We give a version of Kac’s formula for the moment generating function of the total accumulated local time process which enables us to easily obtain the classical first and second Ray–Knight theorems and generalize the second Ray–Knight theorem. In what follows we use the notation  $A^{(1)}$  for the matrix obtained by replacing each of the  $n$  entries in the first column of the  $n \times n$  matrix  $A$  by the number 1.

LEMMA 3.1. *Let  $X$  be a strongly symmetric Markov process with 0-potential density  $u(x, y)$ . Let  $L_\infty$  denote the total accumulated local time of  $X$ . Let  $\Sigma$  be the matrix with elements  $\Sigma_{i, j} = u(x_i, x_j)$ ,  $i, j = 1, \dots, n$  and assume that  $\Sigma$  is invertible. Let  $y = x_1$ . Let  $\Lambda$  be the matrix with elements  $\{\Lambda\}_{i, j} = \lambda_i \delta_{i, j}$ . For all  $\lambda_1, \dots, \lambda_n$  sufficiently small we have*

$$(3.1) \quad E^y \exp \left( \sum_{i=1}^n \lambda_i L_\infty^{x_i} \right) = \frac{\det((I - \Sigma\Lambda)^{(1)})}{\det(I - \Sigma\Lambda)}.$$

PROOF. By Kac’s moment formula,

$$\begin{aligned}
 (3.2) \quad E^y \left( \left( \sum_{j=1}^n \lambda_j L_\infty^{x_j} \right)^k \right) \\
 = k! \sum_{j_1, \dots, j_k=1}^n u(y, x_{j_1}) \lambda_{j_1} u(x_{j_1}, x_{j_2}) \lambda_{j_2} u(x_{j_2}, x_{j_3}) \cdots \\
 u(x_{j_{k-2}}, x_{j_{k-1}}) \lambda_{j_{k-1}} u(x_{j_{k-1}}, x_{j_k}) \lambda_{j_k}
 \end{aligned}$$

for all  $k$ . The proof of this is standard; see, for example, [9], (4.7) or [7].

Let  $\beta = (u(y, x_1)\lambda_1, \dots, u(y, x_n)\lambda_n)$  and  $\bar{1}$  be the transpose of an  $n$ -dimensional vector with all of its elements equal to 1. Observe that  $\sum_{j_k=1}^n u(x_{j_{k-1}}, x_{j_k})\lambda_{j_k}$  is an  $n \times 1$  matrix with entries  $\{\Sigma\Lambda\bar{1}\}_{j_{k-1}}, j_{k-1} = 1, \dots, n$ .

Note also that  $(\Sigma\Lambda)^2\bar{1}$  is an  $n \times 1$  matrix and

$$(3.3) \quad \sum_{j_{k-1}=1}^n u(x_{j_{k-2}}, x_{j_{k-1}})\lambda_{j_{k-1}} \{\Sigma\Lambda\bar{1}\}_{j_{k-1}} = \{(\Sigma\Lambda)^2\bar{1}\}_{j_{k-2}}.$$

Iterating this relationship we get

$$\begin{aligned}
 (3.4) \quad E^y \left( \left( \sum_{j=1}^n \lambda_j L_\infty^{x_j} \right)^k \right) &= k! \beta (\Sigma\Lambda)^{k-1} \bar{1} \\
 &= k! \{(\Sigma\Lambda)^k \bar{1}\}_1,
 \end{aligned}$$

where we use the facts that  $x_1 = y$  and  $\beta(\Sigma\Lambda)^{k-1}$  is an  $n$ -dimensional vector, which is the same as the first row of  $(\Sigma\Lambda)^k$ . It follows from this that

$$\begin{aligned}
 (3.5) \quad E^y \exp \left( \sum_{i=1}^n \lambda_i L_\infty^{x_i} \right) &= \left\{ \sum_{k=0}^\infty (\Sigma\Lambda)^k \bar{1} \right\}_1 \\
 &= \{(I - \Sigma\Lambda)^{-1} \bar{1}\}_1.
 \end{aligned}$$

Equation (3.1) now follows from Cramer’s theorem. This completes the proof of this lemma.  $\square$

REMARK 3.1. The 0-potential  $u(x, y)$  in Lemma 3.1 is positive definite; see, for example, [9], Theorem 3.3. Let  $G = \{G(x); x \in S\}$  be a mean-zero Gaussian process with covariance  $u(x, y)$ . Note that  $(\det(I - \Sigma\Lambda))^{-1}$  is the moment generating function of  $G_n^2/2 + \tilde{G}_n^2/2$ , where  $G_n = (G_{x_1}, \dots, G_{x_n})$ , and similarly for  $\tilde{G}_n$  (see Lemma 2.1). Suppose that we can find a stochastic process  $\{H_x; x \in S\}$  such that  $(\det((I - \Sigma\Lambda)^{(1)}))^{-1}$  is the moment generating function of  $H_{x_1}, \dots, H_{x_n}$ , and suppose this can be done for all finite joint distributions. Then we would have that under the measure  $P_y \times P_H \times P_G \times P_{\tilde{G}}$ ,

$$(3.6) \quad \{L_\infty^x + H_x; x \in S\} = \{G_x^2/2 + \tilde{G}_x^2/2; x \in S\} \quad \text{in law.}$$

**4. A related isomorphism theorem.** In this section we prove an isomorphism theorem closely related to Theorems 1.1 and 1.2 which facilitates their proof. Essentially, for the proofs of Theorems 1.1 and 1.2, we will only have to verify that the conditions of this section apply.

Let  $Z$  is a strongly symmetric Markov process with state space  $S$  and 0-potential density  $v(x, y)$  which can be written in the following form:

$$(4.1) \quad v(x, y) = v_*(x, y) + 1/q,$$

where,  $q > 0$  and  $v_*(x, y)$  is positive definite and satisfies  $v_*(0, y) \equiv 0$  for all  $y \in S$ . Let  $\{\eta_x; x \in S\}$  be the mean-zero Gaussian process with covariance  $v_*(x, y)$ . We have the following isomorphism theorem.

**THEOREM 4.1.** *Let  $Z$  be a strongly symmetric Markov process with 0-potential density  $v(x, y)$  satisfying (4.1), and let  $\{L_\infty^x; x \in S\}$  denote the total accumulated local time of  $Z$ . Then under the measure  $P_0 \times P_n$ ,*

$$(4.2) \quad \{L_\infty^x + (\eta_x + b)^2/2; x \in S\} = \{(\eta_x + \sqrt{2T + b^2})^2/2; x \in S\} \quad \text{in law}$$

for all  $b \in R$ , where  $T$  is an exponential random variable with mean  $1/q$ .

**PROOF.** Set

$$(4.3) \quad \phi_x = \eta_x + G_0,$$

Where  $G_0 \sim N(0, 1/q)$  so that  $\{\phi_x; x \in E\}$  is the mean-zero Gaussian process with covariance function  $v(x, y)$ .

Let  $\Sigma$  be the covariance matrix of  $\phi_{x_1}, \dots, \phi_{x_n}$  where  $x_1 = 0$  and  $x_2, \dots, x_n$  are arbitrary. Note that  $\{\Sigma\}_{i,j} = v(x_i, x_j)$  and in particular  $\{\Sigma\}_{j,1} = v(x_j, 0) = 1/q$ . Consider  $\det((I - \Sigma\Lambda)^{(1)})$ . Subtract the first row of the matrix in this determinant from each of the other rows. Using this row operation it is clear that

$$(4.4) \quad \det((I - \Sigma\Lambda)^{(1)}) = \det(I - \tilde{\Sigma}\tilde{\Lambda}),$$

where  $\tilde{\Sigma}$  is the covariance matrix of  $\eta_{x_2}, \dots, \eta_{x_n}$  and  $\tilde{\Lambda}$  is the natural restriction of  $\Lambda$ , that is,  $\{\tilde{\Lambda}\}_{i,j} = \lambda_{i+1}\delta_{i,j}$ ,  $1 \leq i, j \leq n - 1$ . That is, the reciprocal of (4.4) is the moment generating function of  $(1/2)(\eta_{x_1}^2, \dots, \eta_{x_n}^2) + (1/2)(\tilde{\eta}_{x_1}^2, \dots, \tilde{\eta}_{x_n}^2)$ , where  $\tilde{\eta}$  is an independent copy of  $\eta$ . (Note that we can include  $\eta_{x_1}^2$  since it is equal to 0). Taking into account the comments made in Remark 3.1 we see that we have obtained

$$(4.5) \quad \{L_\infty^x + \eta_x^2/2 + \tilde{\eta}_x^2/2; x \in S\} = \{\phi_x^2/2 + \tilde{\phi}_x^2/2; x \in S\} \quad \text{in law}$$

under the measure  $P_0 \times P_\phi \times P_{\tilde{\phi}}$ .

To obtain (4.2) we note that it follows from Lemma 2.2 that

$$(4.6) \quad \{\phi_x^2/2 + \tilde{\phi}_x^2/2; x \in S\} = \{\tilde{\eta}_x^2/2 + (\eta_x + \sqrt{2T})^2/2; x \in S\} \quad \text{in law,}$$



since by assumption  $h_x = v(x, 0)/v(0, 0) \equiv 1$ . Substituting this in (4.5) we get

$$(4.7) \quad \{L_\infty^x + \eta_x^2/2 + \tilde{\eta}_x^2/2; x \in S\} = \{\tilde{\eta}_x^2/2 + (\eta_x + \sqrt{2T})^2/2; x \in S\} \quad \text{in law,}$$

which implies that

$$(4.8) \quad \{L_\infty^x + \eta_x^2/2; x \in S\} = \{(\eta_x + \sqrt{2T})^2/2; x \in S\} \quad \text{in law.}$$

Since  $T$  is independent of the Gaussian processes it follows from Corollary 2.1 that

$$\begin{aligned} & \{(\eta_x + \sqrt{2T})^2/2 + (\tilde{\eta}_x + b)^2/2; x \in S\} \\ &= \{\eta_x^2/2 + (\tilde{\eta}_x + \sqrt{2T + b^2})^2/2; x \in S\} \quad \text{in law.} \end{aligned}$$

Thus we see that adding  $(\tilde{\eta}_x + b)^2/2$  to each side of (4.8) gives us (4.2).  $\square$

**5. Proofs of Theorems 1.1 and 1.2.** Let  $X$  be a strongly symmetric Markov process. Two Markov processes determined by  $X$  play an important role in these proofs. The first is the process  $Y = \{Y_t; t \in R^+\}$ , which is  $X$  killed when it first hits 0. This process is introduced prior to the statement of Theorem 1.1. To define the second, let  $T = T(q)$  be an exponential random variable with mean  $1/q$ . We define  $Z = \{Z_t; t \in R^+\}$  by

$$(5.1) \quad Z_t = \begin{cases} X_t, & \text{if } t < \tau(T), \\ \Delta, & \text{otherwise,} \end{cases}$$

where  $\tau(t)$  is defined in (1.4).

Recall that  $u_{\{0\}}(x, y)$  denotes the 0-potential density of  $Y$ ; consequently  $u_{\{0\}}(0, y) \equiv 0$ , for all  $y \in S$ .

LEMMA 5.1. *Both  $Y$  and  $Z$  are strongly symmetric Markov processes. If  $X$  is recurrent and  $\tilde{u}(x, y)$  denotes the 0-potential density of  $Z$ , then*

$$(5.2) \quad \tilde{u}(x, y) = u_{\{0\}}(x, y) + 1/q.$$

*Furthermore, both  $u_{\{0\}}(x, y)$  and  $\tilde{u}(x, y)$  are finite, symmetric and positive definite.*

PROOF. The fact that  $Y$  is strongly symmetric follows easily from Hunt's switching identity; see, for example, [9], Section 3, where the reader will also find a proof that  $u_{\{0\}}(x, y)$  is positive definite. [Although the results of that reference are stated for the process killed on existing a compact set, the proofs hold just as well for  $u_{\{0\}}(x, y)$ .] The finiteness of  $u_{\{0\}}(x, y)$  follows from the finiteness of  $u^1(x, y)$  and the well-known fact that  $L_{T_0}^x$  is exponential under  $P^x$ , hence  $L_{T_0}^x$  would be infinite a.s. if  $u_{\{0\}}(x, x) = E^x(L_{T_0}^x)$  were infinite. The strong symmetry of  $Z$  follows from more general results in [12] or [14]. Finally,

to prove (5.2) we use the fact that  $\tau(T) = T_0 + \tau(T) \circ \theta_{T_0}$  and the strong Markov property at  $T_0$  to see that

$$\begin{aligned}
 \tilde{u}(x, y) &= E^x(L_{\tau(T)}^y) = E^x(L_{T_0}^y) + E^0(L_{\tau(T)}^y) \\
 (5.3) \qquad &= E^x(L_{T_0}^y) + E^y(L_{\tau(T)}^0) = E^x(L_{T_0}^y) + E^y(T) \\
 &= u_{\{0\}}(x, y) + 1/q,
 \end{aligned}$$

where the third equality uses the symmetry of  $\tilde{u}(x, y)$  in the form

$$E^0(L_{\tau(T)}^y) = \tilde{u}(0, y) = \tilde{u}(y, 0) = E^y(L_{\tau(T)}^0).$$

This completes the proof of Lemma 5.1.  $\square$

PROOF OF THEOREM 1.1. Given  $X$ , with local time process  $\{L_t^x; x \in S\}$ , we consider the associated Markov process  $Z$  defined in (5.1) with 0-potential  $\tilde{u}(x, y)$ . By Lemma 5.1,  $Z$  satisfies the conditions of Theorem 4.1. Also, the total accumulated local time process for  $Z$  is precisely  $\{L_{\tau(T)}^x; x \in S\}$ . Hence by Theorem 4.1, under the measure  $P_0 \times P_\eta$ ,

$$(5.4) \quad \{L_{\tau(T)}^x + (\eta_x + b)^2/2; x \in S\} = \{(\eta_x + \sqrt{2T + b^2})^2/2; x \in S\} \quad \text{in law}$$

for all  $b \in R$ . (1.6) follows from this by taking the Laplace transform.  $\square$

PROOF OF THEOREM 1.2. Suppose now that the Markov process  $X$  is transient. Let  $u(x, y)$  denote its 0-potential, which is finite. Under  $P^0$ , the total accumulated local time of  $X$ ,  $L_\infty^0$  has an exponential distribution with mean  $u(0, 0)$ . Let  $\rho$  to be an independent exponential random variable with mean  $u(0, 0)$ . Consider the Markov processes  $Y$  and  $Z$  defined in (5.1), for the transient process  $X$ .

LEMMA 5.2. *The potential density of the Markov process  $Z$  with respect to  $m$  is given by*

$$(5.5) \qquad \tilde{u}(x, y) = u_{\{0\}}(x, y) + h_x h_y \frac{1}{q'}$$

where  $h_x = P^x\{T_0 < \infty\} = (u(x, 0)/u(0, 0))$  and  $1/q' = E(T \wedge \rho)$ . Both  $\tilde{u}(x, y)$  and  $u_{\{0\}}(x, y)$  are symmetric and positive definite and  $u_{\{0\}}(0, y) \equiv 0$ , for all  $y \in S$ .

PROOF. The fact that the potential of  $Z$  is a absolutely continuous with respect to  $m$ , is symmetric and positive definite is the content of Lemma 5.1. Let  $\tilde{u}(x, y)$  be its density with respect to  $m$ . Then

$$\begin{aligned}
 \tilde{u}(x, y) &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}\tilde{u}(0, y) \\
 &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}\tilde{u}(y, 0) \\
 &= u_{\{0\}}(x, y) + P^x\{T_0 < \infty\}P^y\{T_0 < \infty\}\tilde{u}(0, 0),
 \end{aligned}$$

where the second equality follows from the symmetry of  $\tilde{u}$ . We now recall that  $\tilde{u}(0, 0) = E^0(L_{\tau(T)}^0)$ , and that  $L_{\tau(T)}^0$  is equal to the minimum of the two exponential random variables,  $T$  and  $L_\infty^0$ . This completes the proof of Lemma 5.2.  $\square$

Note that Lemma 5.2 is actually a generalization of Lemma 5.1 since when  $X$  is recurrent  $h_x = P^x\{T_0 < \infty\} \equiv 1$ .

PROOF OF THEOREM 1.2 (CONTINUED). Let  $A_t$  be a continuous additive functional of  $X$  and let  $R_A(\omega) := \inf\{t|A_t(\omega) > 0\}$ . We call  $A_t$  a local time of  $X$  at  $x \in S$  if  $P^x(R_A = 0) = 1$ , and  $P^y(R_A = 0) = 0$  for all  $y \neq x$ . If  $A_t$  and  $B_t$  are two local times for  $X$  at  $x \in S$ , then  $A_t = cB_t$  for some constant  $c > 0$ . It is easy to check that  $L_t^x$  defined in (1.3) is a local time of  $X$  at  $x$  with potential

$$(5.6) \quad E^y \left( \int_0^\infty e^{-\alpha t} dL_t^x \right) = u^\alpha(y, x) \quad \forall \alpha > 0.$$

We call  $L_t^x$  the normalized local time of  $X$  at  $x$ . However, note that for any real-valued function  $g(x) > 0$ ,  $x \in S$ ,  $g(x)L_t^x$  is also a local time for  $X$  at  $x$ , which we call a nonnormalized local time of  $X$  at  $x$  [unless  $g(x) \equiv 1$ ].

Dividing (5.5) by  $h_x h_y$  we get

$$(5.7) \quad \frac{\tilde{u}(x, y)}{h_x h_y} = \frac{u_{\{0\}}(x, y)}{h_x h_y} + \frac{1}{q'}.$$

It is easy to see that the function  $h_x$  is excessive for  $X, Y$  and  $Z$ . Let  $X^h$  be an  $h$ -path transform of  $X$  taken with respect to  $h_x$  and let  $P^{x/h}$  denote the measure corresponding to  $X^h$ . Denote by  $Y^h$  and  $Z^h$  the corresponding  $h$ -path transforms of  $Y$  and  $Z$ . Note that  $(u(x, y)/h_x h_y)$ ,  $(u_{\{0\}}(x, y)/h_x h_y)$  and  $\tilde{u}(x, y)/h_x h_y$  are the potential densities of  $X^h$ ,  $Y^h$  and  $Z^h$ , respectively, all with respect to  $h_x^2 m(dx)$ . Clearly, they are all symmetric. Let  $\widehat{L}_t^y$  be the normalized local time at  $y$  of  $X^h$  with respect to  $h_x^2 m(dx)$ , so that

$$(5.8) \quad E^{x/h} \int_0^\infty e^{-\alpha t} d\widehat{L}_t^y = \frac{u^\alpha(x, y)}{h_x h_y}.$$

Consider  $\{\eta_x/h_x; x \in S\}$ . This is a mean-zero Gaussian process with covariance  $(u_{\{0\}}(x, y)/h_x h_y)$ . By (5.7),  $Z^h$  satisfies the conditions of Theorem 4.1. Also, the total accumulated local time process for  $Z^h$  is precisely  $\{\widehat{L}_{\tau(T)}^x; x \in S\}$ . Hence by Theorem 4.1, under  $P^{0/h} \times P_\eta$ ,

$$(5.9) \quad \left\{ \widehat{L}_{\tau(T)}^x + \frac{(\eta_x/h_x + b)^2}{2}; x \in S \right\} \\ = \left\{ \frac{(\eta_x/h_x + \sqrt{2V + b^2})^2}{2}; x \in S \right\} \quad \text{in law,}$$

where  $V$  is an exponential random variable with parameter  $q'$  and  $T$  is an exponential random variable with parameter  $q$ . Equivalently, under  $P^{0/h} \times P_\eta$ ,

$$(5.10) \quad \left\{ \tilde{L}_{\tau(T)}^x + \frac{(\eta_x + h_x b)^2}{2}; x \in S \right\} = \left\{ \frac{(\eta_x + h_x \sqrt{2V + b^2})^2}{2}; x \in S \right\} \quad \text{in law,}$$

where  $\tilde{L}_t^x = h_x^2 \widehat{L}_t^x$ . Then  $\tilde{L} := \{\tilde{L}_t^x; x \in S\}$  is a nonnormalized local time for  $X^h$ . Its potential with respect to the process  $X^h$  is

$$(5.11) \quad E^{x/h} \int_0^\infty e^{-\alpha t} h^2(y) d\widehat{L}_t^y = \frac{u^\alpha(x, y) h_y^2}{h_x h_y} = \frac{u^\alpha(x, y) h_y}{h_x},$$

which is the  $\alpha$ -potential of  $X^h$  with respect to  $m$ . This is precisely the potential of  $L$  with respect to the process  $X^h$ , where  $L := \{L_t^x; x \in S\}$  is the normalized local time for  $X$ . Since  $X^h$  is  $X$  killed at the last exit from 0, conditioned to hit 0 after time 0, we see that  $\tilde{L}_t^x = L_t^x$  until the last exit of  $X$  from 0. Let  $\tau_t^- := \inf\{s: L_s^0 \geq t\}$ . It follows that  $\tilde{L}_{\tau(T)}^x = L_{\tau^-(T \wedge L_\infty^0)}^x$ . Therefore (5.10) is equivalent to the statement that under  $P^0 \times P_\eta$ ,

$$\left\{ L_{\tau^-(T \wedge L_\infty^0)}^x + \frac{(\eta_x + h_x b)^2}{2}; x \in S \right\} = \left\{ \frac{(\eta_x + h_x \sqrt{2(T \wedge \rho) + b^2})^2}{2}; x \in S \right\} \quad \text{in law,}$$

where  $\rho$  is an exponentially distributed random variable with mean  $u(0, 0)$  independent of  $T$ . Since this is the Laplace transform of (1.7), the proof of Theorem 1.2 is complete.  $\square$

**6.  $u_{\{0\}}(x, y)$  for symmetric Lévy processes.** To apply Theorems 1.1 and 1.2 one must obtain the 0-potential density  $u_{\{0\}}(x, y)$ , of the process  $X$  killed at  $T_0 = \inf\{s: X_s = 0\}$ , in order to identify the Gaussian process  $\eta$ , defined in (1.5). In this section we do this when  $X$  is a real-valued symmetric Lévy processes.

Let  $X$  be a real-valued symmetric Lévy process defined by

$$(6.1) \quad E e^{i\lambda X_t} = e^{-t\psi(|\lambda|)}.$$

We assume that  $X$  has a local time, which is equivalent to the condition that  $1/(1 + \psi(|\lambda|)) \in L^1$ . It follows simply from (6.1) that the  $\alpha$ -potential density of  $X$ ,

$$(6.2) \quad u^\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\lambda(x - y))}{\alpha + \psi(\lambda)} d\lambda.$$

Since, in this case,  $u^\alpha(x, y)$  depends only on  $|x - y|$ , we also write it as  $u^\alpha(x - y)$ .

In our presentation we distinguish between whether  $X$  is transient or recurrent. Consider the integral

$$(6.3) \quad \int_0^1 \frac{1}{\psi(\lambda)} d\lambda.$$

The Lévy process  $X$  is transient when this integral is finite and is recurrent when this integral is infinite.

**THEOREM 6.1.** *When  $X$  is recurrent,*

$$(6.4) \quad u_{\{0\}}(x, y) = \phi(x) + \phi(y) - \phi(x - y),$$

where

$$(6.5) \quad \phi(x) := \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\psi(\lambda)} d\lambda, \quad x \in R.$$

Consequently, the Gaussian process  $\eta$  defined in (1.5) is such that  $\eta(0) = 0$  and

$$(6.6) \quad E(\eta(x) - \eta(y))^2 = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda(x - y))}{\psi(\lambda)} d\lambda.$$

When  $X$  is transient we can define the stationary Gaussian process  $\{G_x, x \in R\}$  as the process with covariance  $u(x, y) := u^0(x, y)$ , given in (6.2). Let  $\rho_x$  be the projection of  $G_x$  in the subspace of  $L^2$  orthogonal to  $G_0$ . That is, we write  $G_x = \rho_x + h_x G_0$ , where  $h_x = P^x\{T_0 < \infty\} = u(x, 0)/u(0, 0)$  and  $\rho_x = G_x - h_x G_0$ . Thus  $\rho_x$  and  $h_x G_0$  are independent Gaussian process. In this case,

$$(6.7) \quad u_{\{0\}}(x, y) = E(\rho_x \rho_y)$$

$$(6.8) \quad = u(x, y) - \frac{u(x, 0)u(y, 0)}{u(0, 0)}.$$

**PROOF.** It follows from (6.2) that

$$(6.9) \quad \begin{aligned} u^\alpha(0) - u^\alpha(x) &= \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\alpha + \psi(\lambda)} d\lambda \\ &:= \phi_\alpha(x). \end{aligned}$$

Consequently,

$$(6.10) \quad \begin{aligned} \lim_{\alpha \rightarrow 0} (u^\alpha(0) - u^\alpha(x)) &= \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} d\lambda \\ &= \phi(x). \end{aligned}$$

Using the Markov property at time  $T_0$  shows that

$$\begin{aligned}
 u^\alpha(x, y) &= E^x \left( \int_0^\infty e^{-\alpha t} dL_t^y \right) \\
 (6.11) \qquad &= E^x \left( \int_0^{T_0} e^{-\alpha t} dL_t^y \right) + E^x \left( \int_{T_0}^\infty e^{-\alpha t} dL_t^y \right) \\
 &= u_{\{0\}}^\alpha(x, y) + E^x(e^{-\alpha T_0})u^\alpha(0, y).
 \end{aligned}$$

It follows from (6.11) that

$$\begin{aligned}
 (6.12) \qquad &u_{\{0\}}^\alpha(x, y) - u_{\{0\}}^\alpha(x, 0) \\
 &= u^\alpha(x - y) - u^\alpha(x) - \{u^\alpha(y) - u^\alpha(0)\}E^x(e^{-\alpha T_0}) \\
 &= \phi_\alpha(x) + \phi_\alpha(y)E^x(e^{-\alpha T_0}) - \phi_\alpha(x - y).
 \end{aligned}$$

The assumption that  $X$  is recurrent means that  $P^x(T_0 < \infty) = 1$ . Consequently,  $\lim_{\alpha \rightarrow 0} E^x(e^{-\alpha T_0}) = 1$ . Also, clearly,  $u_{\{0\}}^\alpha(x, y)$  tends to  $u_{\{0\}}(x, y)$  as  $\alpha$  decreases to zero. Therefore we can take the limits in (6.12) and use the fact that  $u_{\{0\}}^\alpha(x, y) = 0$  to obtain (6.4). Since

$$(6.13) \qquad E(\eta(x) - \eta(y))^2 = u_{\{0\}}(x, x) + u_{\{0\}}(y, y) - 2u_{\{0\}}(x, y)$$

we get (6.6).

Now suppose that  $X$  is transient. Since the 0-potential of  $X$  exists in this case we can simply take the limit  $\alpha \rightarrow 0$  in (6.11) to get

$$(6.14) \qquad u_{\{0\}}(x, y) = u(x, y) - p^x\{T_0 < \infty\}u^\alpha(0, y).$$

Using  $P^x\{T_0 < \infty\} = u(x, 0)/u(0, 0)$  we get (6.7).  $\square$

**REMARK 6.1.** When  $X$  is a symmetric stable process with index  $\beta \in (1, 2]$ ,  $\psi(|\lambda|) = |\lambda|^\beta$ . Using this in (6.5) and making the change of variables  $\lambda|x - y| = s$ , we see that

$$(6.15) \qquad E(\eta(x) - \eta(y))^2 = c_\beta|x - y|^{\beta-1},$$

where

$$(6.16) \qquad c_\beta = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos s}{s^\beta} ds.$$

**7. First Ray-Knight theorem.** We show that the first Ray-Knight theorem is an immediate consequence of our version of Kac’s moment formula.

**THEOREM 7.1 (First Ray-Knight theorem).** *Let  $B$  be Brownian motion on  $R^+$ , starting from  $y > 0$  and let  $\{L_t^x; (x, t) \in R^+ \times R^+\}$  denote its local time process. Let  $T_0$  denote the first hitting time of 0. Then, under the measure  $P^y \times \tilde{P}_1^0 \times \tilde{P}_2^0$ ,*

$$(7.1) \qquad \{L_{T_0}^x; 0 \leq x \leq y\} = \{\tilde{B}_{1,x}^2 + \tilde{B}_{2,x}^2; 0 \leq x \leq y\} \quad \text{in law,}$$

where  $\{\tilde{B}_{i,x}; x \geq 0\}$ ,  $i = 1, 2$  are independent real-valued Brownian motions with measures  $\tilde{P}_i^0$ , independent of the original Brownian motion  $B$ .

PROOF. We apply Lemma 3.1 to the process  $Y$  which is Brownian motion, starting at  $y > 0$  and killed the first time it hits zero. In this case  $L_{T_0}$  is the total accumulated local time of  $Y$ . The 0-potential of  $Y$  is  $u_{\{0\}}(x, y) = 2(|x| \wedge |y|)$  when  $xy \geq 0$  and  $u_{\{0\}}(x, y) = 0$  and when  $xy < 0$ . This is well known and is also included in Theorem 6.1. We take  $y = x_1 > \dots > x_n > 0$  and  $\Sigma_{i,j} = u_{\{0\}}(x_i, x_j)$  in Lemma 3.1. In the next paragraph we show that  $\det((I - \Sigma\Lambda)^{(1)}) = 1$ , which implies that  $H_x$  in Remark 3.1 is identically zero. Consequently, (7.1) follows from (3.6) since the Gaussian process  $G$  in (3.6), with covariance  $2(x \wedge y)$  for  $x, y > 0$ , is equal in law to  $\sqrt{2}B$ .

Let  $D$  be the matrix obtained by subtracting the first row of  $(I - \Sigma\Lambda)^{(1)}$  from each of the other rows. Clearly,  $D_{j,j} = 1$ ,  $j = 1, \dots, n$ ;  $D_{j,1} = 0$ ,  $j = 2, \dots, n$ ; and  $D_{j,k} = 0$ ,  $k = j+1, \dots, n$ . This shows that the matrix  $\tilde{D} := \{D_{i,j}\}_{i,j=2}^n$  is a lower triangular matrix; that is, all its entries above the diagonal are equal to zero. Thus  $\det((I - \Sigma\Lambda)^{(1)}) = \det(D) = D_{1,1} \det(\tilde{D}) = 1$ . This completes the proof of this theorem.  $\square$

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## REFERENCES

- [1] BASS, R., EISENBAUM, N. and SHI, Z. (2000). The most visited sites of symmetric stable processes. *Probab. Theory Related Fields* **116** 391–404.
- [2] BERTOIN, J. (1996). *Lévy Processes*. Cambridge Univ. Press.
- [3] DYNKIN E. B. (1983). Local times and quantum fields. In *Seminar on Stochastic Processes 1982* (E. Çinlar, K. L. Chung and R. K. Gettoor, eds.) 69–84. Birkhäuser, Boston.
- [4] EISENBAUM, N. (1994). Dynkin's isomorphism theorem and the Ray–Knight theorems. *Probab. Theory Related Fields* **99** 321–335.
- [5] EISENBAUM, N. (1997). Théorèmes limites pour les temps locaux d'un processus stable symétrique. *Séminaire de Probabilités XXXI. Lecture Notes in Math.* **1655** 216–224. Springer, Berlin. (Errata: *Séminaire de Probabilités XXXIII. Lecture Notes in Math.* **1709** 417).
- [6] EISENBAUM, N. and KASPI, H. (1993). A necessary and sufficient condition for the Markov property of the local time process. *Ann. Probab.* **21** 1591–1598.
- [7] FITZSIMMONS, P. and PITMAN, J. H. (1999). Kac's moment formula and the Feynman–Kac formula for additive functionals of a Markov process. *Stochastic Process Appl.* **79** 117–134.
- [8] MARCUS, M. B. (2000). The most visited sites of certain Lévy processes. *J. Theoret. Probab.* To appear.
- [9] MARCUS, M. B. and ROSEN, J. (1992). Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab.* **20** 1603–1684.
- [10] MARCUS, M. B. and ROSEN, J. (1996). Gaussian chaos and sample path properties of additive functionals of symmetric Markov processes *Ann. Probab.* **24** 1130–1177.
- [11] MARCUS, M. B. and ROSEN J. (2000). Gaussian processes and the local times of symmetric Lévy processes. In *Lévy Processes and Their Applications* (O. Barndorff-Nielsen, T. Mikosch and S. Resnick, eds.) Birkhauser, Boston. To appear.

- [12] MITRO, J. B. (1979). Dual Markov functionals: applications of a useful auxiliary process. *Z. Wahrsch. Verw. Gebiete* **48** 976–114.
- [13] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. Springer, Berlin.
- [14] WALSH, J. B. (1972). Markov processes and their functionals in duality. *Z. Wahrsch. Verw. Gebiete* **24** 229–246.

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