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ADDITIVE FUNCTIONALS OF SEVERAL LÉVY PROCESSES AND INTERSECTION LOCAL TIMES[1]

BY MICHAEL B. MARCUS AND JAY ROSEN[2]

City College of CUNY and College of Staten Island, CUNY

Different extensions of an isomorphism theorem of Dynkin are developed and are used to study two distinct but related families of functionals of Lévy processes; n-fold "near-intersections" of a single Lévy process and continuous additive functionals of several independent Lévy processes. Intersection local times for n independent Lévy processes are also studied. They are related to both of the above families. In all three cases sufficient conditions are obtained for the almost sure continuity of these functionals in terms of the almost sure continuity of associated Gaussian chaos processes. Concrete sufficient conditions are given for the almost sure continuity of these functionals of Lévy processes.

1. Introduction. In this paper we develop different extensions of an isomorphism theorem of Dynkin and use them to study two distinct but related families of functionals of Lévy processes: n-fold "near-intersections" of a single Lévy process and continuous additive functionals of several independent Lévy processes. We also study intersection local times for n independent Lévy processes. They are related to both of the above families. In all three cases, sufficient conditions are obtained for the almost sure continuity of these functionals in terms of the almost sure continuity of associated Gaussian chaos processes. Results from [3] are used to give concrete sufficient conditions for the almost sure continuity of these functionals of Lévy processes.

Let $(\Omega, \mathscr{F}(t), X(t), P^x)$ be a strongly symmetric Lévy process in R^d with 1-potential density $u^1(x)$. The definition of a strongly symmetric Markov process is given in [5]. For the purposes of this paper it is enough to note that this means that the Lévy process has a symmetric transition probability density. We also require that the 1-potential density satisfies

$$\int |x|^\alpha u^1(x)\, dx < \infty \tag{1.1}$$

for some $\alpha > 0$. This holds, in particular, if $X(t)$ is in the domain of attraction of a stable process.

In [4] we consider the n-fold intersections of $X = \{X(t), t \in R^+\}$. This entails studying a functional of the form

$$L_{n,\varepsilon}(\mu, t) =_{\text{def}} \int \int_{[0,t]^n} \prod_{j=1}^{n} f_\varepsilon(X(t_j) - y)\, dt_1 \cdots dt_n\, d\mu(y), \tag{1.2}$$

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where f_ε is an approximate δ-function at zero. The idea is to take the limit as $\varepsilon \to 0$. But, when $n > 1$, the limit is infinite. This is dealt with by a process called renormalization, which consists of forming a linear combination of $\{L_{k,\varepsilon}\}_{k=1}^n$ which has a finite limit as $\varepsilon \to 0$. This is done in [4] and requires a considerable amount of complicated analysis.

In different types of intersection problems and in the study of continuous multiply additive functionals of independent Lévy processes, we can analyze functionals similar to $L_{n,\varepsilon}(\mu, t)$ without renormalization. We consider three cases.

CASE 1 (Intersection local times (ILTs) of independent Lévy processes). Let $(\Omega_j, \mathscr{F}_j(t), X_j(t), P_j^x)$, $j = 1, \dots, n$ be independent strongly symmetric Lévy processes on R^d. Let μ be a positive measure on R^d. Define

$$(1.3) \qquad \widetilde{L}_{n,\varepsilon}(\mu, t) =_{\text{def}} \int \int_{[0,t]^n} \prod_{j=1}^n f_\varepsilon(X_j(t_j) - y)\, dt_1 \cdots dt_n\, d\mu(y).$$

When the limit, as $\varepsilon \to 0$, of $\widetilde{L}_{n,\varepsilon}(\mu, t)$ exists for each $x \in R^d$ we think of it as measuring the n-fold intersections, up to time t, of the paths of the n independent Lévy processes X_j, with intersection points weighted by the measure μ. Let μ_x denote translation of the measure μ by x. Note that

$$(1.4) \qquad \widetilde{L}_{n,\varepsilon}(\mu_x, t) = \int \int_{[0,t]^n} \prod_{j=1}^n f_\varepsilon(X_j(t_j) - y - x)\, dt_1 \cdots dt_n\, d\mu(y).$$

When the limit of (1.4) exists in L^2 of the probability space, as $\varepsilon \to 0$, for each $x \in R^d$, we consider the stochastic process

$$(1.5) \qquad \widetilde{L}_n(\mu_x, t) =_{\text{def}} \lim_{\varepsilon \to 0} \widetilde{L}_{n,\varepsilon}(\mu_x, t)$$

on $R^d \times R^+$. In Theorem 1.1 we give a sufficient condition for

$$(1.6) \qquad \{\widetilde{L}_n(\mu_x, t), (x, t) \in R^d \times R^+\}$$

to have a version which is continuous almost surely.

$\widetilde{L}_n(\mu_x, t)$ is a continuous additive functional of several independent Lévy process. We now consider these processes in greater generality.

CASE 2 (Continuous additive functionals (CAFs) of several independent Lévy processes). Let $X_1, \dots, X_n$ be n independent strongly symmetric Lévy processes in R^d, with associated σ-fields $\mathscr{F}_{j,t}$, probabilities P_j^y and translation operators $\theta_{j,s}$. Let $u_j^1(x)$ denote the 1-potential density of X_j. A functional $A_{t_1,\dots,t_n}$ is called a continuous additive functional of the independent Lévy processes X_j, $j = 1, \dots, n$ when it is positive, continuous and increasing in $t_1, \dots, t_n$, measurable with respect to $\bigotimes_{j=1}^n \mathscr{F}_{j,t_j}$ and is separately additive in each variable, that is, when

$$(1.7) \qquad A_{s_1,\dots,s_j+t_j,\dots,s_n} = A_{s_1,\dots,s_j,\dots,s_n} + A_{s_1,\dots,t_j,\dots,s_n} \circ \theta_{j,s_j}.$$

An example of such a continuous additive functional is given by

$$A_{t_1,\dots,t_n} = \int_0^{t_1} \cdots \int_0^{t_n} f(X_1(r_1), \dots, X_n(r_n))\, dr_1 \dots dr_n, \tag{1.8}$$

where f is a positive measurable function. When $A_{t_1,\dots,t_n}$ is a continuous additive functional, we define the "1-potential" of A to be

$$\Big(\mathop{\times}_{j=1}^{n} U_j^1\Big)_A (y_1, \dots, y_n) = E^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n}(A_{\lambda_1,\dots,\lambda_n}), \tag{1.9}$$

where $\lambda_1, \dots, \lambda_n$ are independent mean-1 exponential random variables.

One can show, as in [2], IV.2, that a continuous additive functional is determined by its 1-potential whenever the latter is finite. Here, $E^{y_1,\dots,y_n}_{\lambda_1,\dots,\lambda_n}$ denotes expectation with respect to the product measure $\bigotimes_{j=1}^n P^{y_j}_{j,\lambda_j}$, where for each j, $P^{y_j}_{j,\lambda_j} = P^{y_j}_j \otimes P_{\lambda_j}$, and P_{λ_j} is the probability measure for λ_j.

Let $t = (t_1, \dots, t_n)$ and let μ be a positive measure on $(R^d)^n$. We use $(\times_{j=1}^n L_j)(\mu, t)$ to denote the continuous additive functional with 1-potential,

$$\Big(\mathop{\times}_{j=1}^{n} U_j^1\Big)\mu(y_1, \dots, y_n) = \int \prod_{j=1}^{n} u_j^1(y_j - x_j)\, d\mu(x_1, \dots, x_n), \tag{1.10}$$

whenever such a continuous additive functional exists. In Theorem 1.2 we obtain a sufficient condition for the almost sure continuity of

$$\left\{\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\mu_x, t), (x, t) \in (R^d)^n \times (R_+)^n\right\} \tag{1.11}$$

where μ_x denotes translation of μ by $x \in (R^d)^n$.

CASE 3 (n-fold "near-intersections" of a single Lévy process). Let $(\Omega, \mathcal{F}(t), X(t), P^x)$ be a Lévy processes on R^d. Let μ be a positive measure on R^d and let $x = (x_1, \dots, x_n) \in (R^d)^n$. Define

$$L^{n,\varepsilon}(x, t; \mu) = \int \int_{[0,t]^n} \prod_{j=1}^{n} f_\varepsilon(X(t_j) - y - x_j)\, dt_j\, d\mu(y) \tag{1.12}$$

and set

$$L^n(x, t; \mu) = \lim_{\varepsilon \to 0} L^{n,\varepsilon}(x, t; \mu), \tag{1.13}$$

whenever this limit exists.

Heuristically we have

$$L^n(x, t; \mu) = \int \int_{[0,t]^n} \prod_{j=1}^{n} \delta(X(t_j) - y - x_j)\, dt_j\, d\mu(y), \tag{1.14}$$

where δ is the Dirac delta function at zero. Thus $L^n(x, t; \mu)$ measures the amount of "time" when $X(s_i)$ differs from y by x_i for all $i = 1, \dots, n$, with y weighted by μ. When $x_i - x_j$ is close to zero for any $i \neq j$, $L^n(x, t; \mu)$ actually measures "near intersections." When $x_i - x_j$ is close to zero for all $i \neq j$, $1 \le i$, $j \le n$, $L^n(x, t; \mu)$ measures n-fold "near intersections."

In general $L^n(x, t; \mu)$ blows up as the $x_i - x_j \to 0$. In fact, if $x_i = x_j$ for any $i \neq j$, the limit in (1.13) does not exist. In Theorem 1.3 we obtain a sufficient condition for

$$\{L^n(x, t; \mu);\ x \in (R^d)^n_{\neq} \times R_+\} \tag{1.15}$$

to have a version which is continuous almost surely, where

$$(R^d)^n_{\neq} =_{\text{def}} \{x = (x_1, \dots, x_n) \colon x_i \neq x_j,\ \forall\, i, j;\ i \neq j\}. \tag{1.16}$$

Continuity of the three types of processes described above is implied by the continuity of corresponding Gaussian chaos processes, which we now define. Let $X_1, \dots, X_n$ be n independent strongly symmetric Lévy processes in R^d, with 1-potential densities $u^1_j(x)$, $j = 1, \dots, n$. For positive measures μ on R^d let

$$\mathscr{G}^m_j =_{\text{def}} \left\{\int\int (u^1_j(x-y))^m \, d\mu(x)\, d\mu(y) < \infty\right\}. \tag{1.17}$$

We define $\{G_{j,\nu}, \nu \in \mathscr{G}^1_j\}$ to be an independent mean zero Gaussian process with covariance

$$E(G_{j,\theta} G_{j,\phi}) = \int\int u^1_j(x-y)\, d\theta(x)\, d\phi(y) \tag{1.18}$$

for $\theta, \phi \in \mathscr{G}^1_j$. Let f_ε be a smooth approximate identity. That is, $f_\varepsilon(y)$ is a smooth positive symmetric function on $(y, \varepsilon) \in R^d \times (0, 1]$ with support in the ball of radius ε and such that $\int f_\varepsilon(y)\, dy = 1$. Let $\rho_\delta(dy) = f_\delta(y)\, dy$ and $\rho_{\delta, x}(dy) = f_{\delta, x}(y)\, dy$, where for any function $h(y)$ we use the notation $h_x(y) = h(y - x)$. It is easily seen that $\rho_{\delta, x}(dy) \in \mathscr{G}^1_j$. Let

$$G_{j, x, \delta} =_{\text{def}} G_{j, \rho_{\delta, x}} \tag{1.19}$$

and

$$u^1_{j, \delta, \delta'}(x - x') =_{\text{def}} \int\int u^1_j(y - y') f_{\delta, x}(y) f_{\delta', x'}(y')\, dy\, dy'. \tag{1.20}$$

We see that for each j, $G_{j, x, \delta}$ is a mean zero Gaussian process with covariance

$$E(G_{j, x, \delta} G_{j, x', \delta'}) = u^1_{j, \delta, \delta'}(x - x'). \tag{1.21}$$

We define the second-order Gaussian chaos process

$$H_{j, x, \delta} = G^2_{j, x, \delta} - EG^2_{j, x, \delta}. \tag{1.22}$$

Also for positive measures μ on R^d, we define

$$\mathscr{G}^{2, n} = \left\{\mu \,\middle|\, \int\int \left(\prod_{j=1}^n u^1_j(x-y)\right)^2 d\mu(x)\, d\mu(y) < \infty\right\}. \tag{1.22a}$$

and for positive measures μ on $(R^d)^n$ we define

$$\mathscr{G}_n^{2,n} = \left\{\mu \,\middle|\, \int\int\left(\prod_{j=1}^n u_j^1(x_j - y_j)\right)^2 d\mu(x_1,\dots,x_n)\, d\mu(y_1,\dots,y_n) < \infty\right\}. \tag{1.23}$$

Corresponding to the processes defined in Cases 1–3, we define three different classes of Gaussian chaos processes.

1′. Let $x \in R^d$ and let $\mu \in \mathscr{G}^{2,n}$,

$$\mathscr{H}_1(x;\mu) =_{\text{def}} \lim_{\delta\to 0}\int \prod_{j=1}^n H_{j,\,x+y,\,\delta}\, d\mu(y). \tag{1.24}$$

This limit exists in L^2 of the probability space. Note that $\mathscr{H}_1(x;\mu) = \mathscr{H}_1(0;\mu_x)$; that is, the dependence of $\mathscr{H}_1(x;\mu)$ on x is only through the translated measure μ.

2′. Let $x \in (R^d)^n$ and write it as $(x_1,\dots,x_n)$. Let $\mu \in \mathscr{G}_n^{2,n}$,

$$\mathscr{H}_2(x;\mu) =_{\text{def}} \lim_{\delta\to 0}\int \prod_{j=1}^n H_{j,\,x_j+y_j,\,\delta}\, d\mu(y). \tag{1.25}$$

This limit exists in L^2 of the probability space. Note that $\mathscr{H}_2(x;\mu) = \mathscr{H}_2(0;\mu_x)$.

3′. Let $\mu \in \mathscr{G}^{2n}$ and note that by the Schwarz inequality $\mathscr{G}^r \subset \mathscr{G}^{2n}$, for all $r \le 2n$. Let $\{G_{x,\delta}^{(j)}\}_{j=1}^{2n}$ be independent copies of $G_{1,x,\delta}$.

$$\mathscr{H}_{3,r}(x;\mu) =_{\text{def}} \lim_{\delta\to 0}\int \prod_{j=1}^r G_{x_j+y,\,\delta}^{(j)}\, d\mu(y), \qquad x \in (R^d)^r. \tag{1.26}$$

This limit exists in L^2 of the probability space. Note that $\mathscr{H}_{3,r}(x;\mu) \ne \mathscr{H}_{3,r}(0;\mu_x)$. In fact μ_x makes no sense. The measure μ is on R^d, whereas $x \in (R^d)^r$.

THEOREM 1.1 (Continuity theorem for ILTs). *Let $\mu \in \mathscr{G}^{2,n}$. If $\{\mathscr{H}_1(x;\mu), x \in R^d\}$ is continuous almost surely then $\{\widetilde{L}_n(\mu_x,t);\,(x,t) \in R^d \times R_+\}$ is continuous almost surely.*

THEOREM 1.2 (Continuity theorem for CAFs). *Let $\mu \in \mathscr{G}_n^{2,n}$. If $\{\mathscr{H}_2(x;\mu), x \in (R^d)^n\}$ is continuous almost surely then $\{(\times_{j=1}^n L_j)(\mu_x, t_1,\dots,t_n);\,(x,t_1,\dots,t_n) \in (R^d)^n \times (R_+)^n\}$ is continuous almost surely.*

THEOREM 1.3 (Continuity theorem for "near intersections"). *Let $\mu \in \mathscr{G}^{2n}$ and assume that for each $1 \le r \le 2n$, $\{\mathscr{H}_{3,r}(x;\mu),\, x \in (R^d)^r\}$ is continuous almost surely. Then $\{L^n(x,t;\mu),\,(x,t) \in (R^d)^n_{\ne} \times R^+\}$ is continuous almost surely.*

Let $x = (x_1, \dots, x_n) \in (R^d)^n$ and similarly for y. In the course of proving Theorem 1.2 we show that the limit, as $\varepsilon \to 0$, of

$$\int \left\{ \prod_{j=1}^{n} \int_0^{t_j} f_\varepsilon(X_j(s_j) - y_j)\, ds_j \right\} d\mu_x(y) \tag{1.27}$$

is continuous almost surely on $(R^d)^n \times (R_+)^n$. Comparing this with (1.3) we see that, by taking μ in Theorem 1.2 to be supported on the diagonal of $(R^d)^n$, Theorem 1.1 follows from Theorem 1.2 under the possibly stronger hypothesis that $\{\mathscr{H}_2(x;\mu),\ x \in (R^d)^n\}$ is continuous almost surely. The proof of Theorem 1.1, as stated, is exactly the same as the proof of Theorem 1.2 but with $x \in R^d$ instead of $(R^d)^n$.

The processes in Case 1 are also related to those in Case 3 in the sense that the examples given in Examples 1.1 below are valid for both the continuity of (1.15) and the continuity of (1.6). This is because our methods for obtaining simplified sufficient conditions depend on the support of the measure μ. In both (1.6) and (1.15) the measure is supported on R^d, whereas in (1.11) it is supported on $(R^d)^n$.

We now use Theorems 1.1–1.3 to obtain concrete sufficient conditions for the continuity of the processes in (1.6), (1.11) and (1.15) in terms of $\{u_j^1\}_{j=1}^n$ and μ. We do this by finding sufficient conditions for the continuity of the chaoses $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$. This is done by showing that the continuity of each of these chaoses is controlled by the continuity of a related decoupled mean zero Gaussian chaos. We use a well-known sufficient condition for the continuity of Gaussian chaoses.

Let $\{\mathscr{I}_m(x), x \in T\}$ be a mean zero mth order Gaussian chaos process on T, where T is some index set. For $x, y \in T$ set

$$\tau_m(x, y) = \Big(E\big(\mathscr{I}_m(x) - \mathscr{I}_m(y)\big)\Big)^{1/2}. \tag{1.28}$$

A sufficient condition for $\{\mathscr{I}_m(x), x \in T\}$ to be continuous almost surely is that

$$\int_0^D (\log N_{\tau_m}(T, \varepsilon))^{m/2}\, d\varepsilon < \infty, \tag{1.29}$$

where $N_{\tau_m}(T, \varepsilon)$ is the minimum number of balls of radius ε, in the metric τ_m, that covers T, and D is the diameter of T with respect to τ_m. Here $N_{\tau_m}(T, \varepsilon)$ is called the metric entropy of T with respect to τ_m.

One can readily obtain τ_m for decoupled chaos processes but, generally, it is complicated and we don't know how to make sense of what condition (1.29) means in terms of $\{u_j^1\}_{j=1}^n$ and μ. In [3], with certain smoothness conditions imposed on the 1-potentials or on the measures μ, sufficient conditions are given for (1.29) that are easy to understand. We use these conditions in Theorems 1.4–1.6. In Section 6 we show how they follow from the results of [3].

In Theorems 1.4 and 1.5 we use the condition that the 1-potentials are in Class A. We define this class as follows: let $h\colon R^q \to R^1$ and $b \in R^q$. Define $\widetilde{\Delta}_b h(s) = h(s+b/2) - h(s-b/2)$ and $\widetilde{\Delta}^2_{b,c} h(s) = \widetilde{\Delta}_b \widetilde{\Delta}_c h(s)$. (We use $\widetilde{\Delta}$ to denote symmetric difference.) We say u^1 belongs to Class A if it is radially symmetric and $u^1(|s|)$ is regularly varying at the origin, u^1 is bounded away from the origin and there exists an $s_0 > 0$ such that for $|s| \le s_0$,

$$|\widetilde{\Delta}_b u^1(s)| \le C|b|\frac{u^1(s)}{|s|} \quad \text{for } |b| \le \frac{|s|}{4} \tag{1.30}$$

and

$$|\widetilde{\Delta}^2_{b,c} u^1(s)| \le C(|b||c|)\frac{u^1(s)}{|s|^2} \quad \text{for } |b|, |c| \le \frac{|s|}{4}. \tag{1.31}$$

Also, if $u^1(|s|)$ is slowly varying at the origin we require that it is asymptotic to a decreasing function at the origin. This condition is clearly satisfied when $u^1(|s|)$ is regularly varying at the origin with index less than zero. Let $|s| = r$ and consider $(u^1(r))'$, the derivative of u^1 with respect to r. We require that for all $r_0 > 0$, $(u^1(r))' \vee (u^1(r))'' \le C_{r_0}$ for all $r \ge r_0 > 0$, where C_{r_0} is a constant depending only on r_0.

Class A includes the 1-potentials of radially symmetric stable processes, including Brownian motion as well as Lévy processes in their domains of attraction. In [4] we show that the 1-potentials of a large class of symmetric Lévy processes, which we call "stable mixtures," are in Class A.

Since we are interested in "several Lévy processes" or in "intersections," we are really only concerned when $n \ge 2$ in Theorems 1.1–1.3. The conditions that $\mu \in \mathscr{G}^{2n}$, $\mathscr{G}^{2,n}$ or $\mathscr{G}^{2,n}_n$, for $n \ge 2$ restricts the Lévy processes to which our results apply. When $\mu \in \mathscr{G}^{2n}$ or $\mathscr{G}^{2,n}$ we can only consider the 1-potentials of Lévy processes on R^d for $d = 1$ or 2. Furthermore, for stable processes, when $\mu \in \mathscr{G}^{2n}$, the index of stability, say α, must satisfy $\alpha > d(1 - 1/2n)$, except for Brownian motion in R^2 and the symmetric Cauchy process in R^1, for which there exist measures $\mu \in \mathscr{G}^{2n}$ for all n. Also, since we ask that $u^1(0) = \infty$, in R^1 we only consider stable processes with index $\alpha \le 1$.

The remarks in the preceding paragraph all follow from that fact that, if (1.17), (1.22a) or (1.23) hold, they must hold for Lebesgue measure on $[0, 1]^d$. This means that if $\mu \in \mathscr{G}^{2n}$ we must have

$$\int_{[0,1]^d} \left(u^1(x)\right)^{2n} dx < \infty. \tag{1.32}$$

If $\mu \in \mathscr{G}^{(2,n)}$ we must have

$$\int_{[0,1]^d} \prod_{j=1}^n \left(u^1_j(x)\right)^2 dx < \infty \tag{1.33}$$

and if $\mu \in \mathscr{G}^{(2,n)}_n$ we must have that

$$\int_{[0,1]^d} \left(u^1_j(x)\right)^2 dx < \infty, \qquad \forall\, 1 \le j \le n. \tag{1.34}$$

We see from (1.34) that in order that $\mu \in \mathscr{G}^{2,n}_n$ we can only consider the 1-potentials of Lévy processes on R^d for $d = 1, 2$ or 3. Furthermore, considering only stable processes, we have that when $\mu \in \mathscr{G}^{2,n}$, the index of stability α, must satisfy $\alpha > 1$ when $d = 2$ and $\alpha > 3/2$ when $d = 3$. As above, in R^1 we only consider stable processes with index $\alpha \le 1$.

We now give sufficient conditions for the continuity of the processes in Cases 1–3 that are easy to compute.

THEOREM 1.4 (Continuity of "near intersections"). *Let* $\mu \in \mathscr{G}^{2n}$. *Then* $\{L^n(x, t; \mu), (x, t) \in (R^d)^n_{\neq} \times R^+\}$ *is continuous almost surely if*:

(i) μ *is in Class* A *and*

$$\int\int_{|x-y|\le\varepsilon} \big(u^1(x-y)\big)^{2n} (\log 1/|x-y|)^{(2n+\delta)}\, d\mu(x)\, d\mu(y) < \infty \tag{1.35}$$

for any $\varepsilon, \delta > 0$.

(ii) *Without requiring that* μ *is in Class* A,

$$|\hat{\mu}(\xi)| = O\bigg(\frac{1}{\mathscr{F}[(u^1)^{2n}](\xi)|\xi|^d(\log|\xi|)^{2n+1+\delta}}\bigg)^{1/2}, \tag{1.36}$$

$\xi \in R^d$, *for some* $\delta > 0$, *where* $\mathscr{F}[\cdot]$ *denotes Fourier transform.*

Note that (1.35) holds with respect to Lebesgue measure on $[0, 1]^d$ if and only if

$$\int_{[0,1]^d} \big(u^1(x)\big)^{2n} (\log 1/|x|)^{(2n+\delta)}\, dx < \infty. \tag{1.37}$$

THEOREM 1.5 (Continuity of ILTs). *Let* $\mu \in \mathscr{G}^{2,n}$. *Then* $\{\widetilde{L}^n(\mu_x, t), (x, t) \in R^d \times R^+\}$ *is continuous almost surely if*:

(i) μ *is in Class* A *and*

$$\int\int_{|x-y|\le\varepsilon} \prod_{j=1}^{n} \big(u^1_j(x-y)\big)^2 (\log 1/|x-y|)^{(2n+\delta)}\, d\mu(x)\, d\mu(y) < \infty \tag{1.38}$$

for any $\varepsilon, \delta > 0$.

(ii) *Without requiring that* μ *is in Class* A,

$$|\hat{\mu}(\xi)| = O\bigg(\frac{1}{\mathscr{F}[\prod_{j=1}^n (u^1_j)^2](\xi)|\xi|^d(\log|\xi|)^{2n+1+\delta}}\bigg)^{1/2}, \tag{1.39}$$

$\xi \in R^d$, *for some* $\delta > 0$.

EXAMPLES 1.1. Obviously the conditions for continuity in Theorems 1.4 and 1.5 are the same when all the u^1_j in Theorem 1.5 are equal. In this case the following examples, taken from [3], are examples of 1-potentials u^1 and measures μ for which $\{L^n(x, t; \mu), (x, t) \in (R^d)^n_{\neq} \times R^+\}$ and $\{\widetilde{L}^n(\mu_x, t), (x, t) \in R^d \times R^+\}$ are continuous almost surely. Note that in these examples (1.36), and equivalently, (1.39) are satisfied.

(i) u^1 is the 1-potential of Brownian motion in R^2 and

$$|\hat{\mu}(\xi)| = O\Big(\log^{-(2n+\varepsilon)}(|\xi|)\Big) \quad \text{as } |\xi| \to \infty, \tag{1.40}$$

$\xi \in R^2$, for some $\varepsilon > 0$.

(ii) u^1 is the 1-potential of a symmetric stable process in R^2 of index $2 - 1/n < \alpha < 2$ and

$$|\hat{\mu}(\xi)| = O\Big(|\xi|^{-(2-\alpha)n} \log^{-(n+1/2+\varepsilon)}(|\xi|)\Big) \quad \text{as } |\xi| \to \infty, \tag{1.41}$$

$\xi \in R^2$, for some $\varepsilon > 0$.

(iii) u^1 is the 1-potential of a symmetric stable process in R^1 of index 1 and

$$|\hat{\mu}(\xi)| = O\Big(\log^{-(2n+\varepsilon)}(|\xi|)\Big) \quad \text{as } |\xi| \to \infty, \tag{1.42}$$

$\xi \in R^1$, for some $\varepsilon > 0$.

(iv) u^1 is the 1-potential of a symmetric stable process in R^1 of index $1 - 1/2n < \alpha < 1$ and

$$|\hat{\mu}(\xi)| = O\Big(|\xi|^{-(1-\alpha)n} \log^{-(n+1/2+\varepsilon)}(|\xi|)\Big) \quad \text{as } |\xi| \to \infty, \tag{1.43}$$

$\xi \in R^1$, for some $\varepsilon > 0$.

We next consider sufficient conditions for the almost sure continuity of $\{(\times_{j=1}^n L_j)(\mu_x, t), (x, t) \in (R^d)^n \times (R_+)^n\}$.

We first note that when μ is a product measure the situation simplifies considerably. If $\mu(y) = \mu_1(y_1)\cdots\mu_n(y_n)$ where $y = (y_1, \dots, y_n)$, then

$$\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\mu_x, t) = \prod_{j=1}^{n} L_{j, t_j}^{\mu_{x_j}}, \tag{1.44}$$

where $L_{j,t}^{\nu}$ is the continuous additive functional of X_j with Revuz measure ν.

Let X be a Lévy process with 1-potential u^1 and let L_t^{μ} be the continuous additive functional of X with Revuz measure μ. Sufficient conditions for the continuity almost surely of $\{L_t^{\mu_x}, (x, t) \in R^d \times R_+\}$ can be obtained from Theorem 1.5 with $n = 1$ and examples are given in Examples 1.1. Continuous additive functionals of the type L_t^{μ} are studied extensively in [6]. In some cases significantly sharper results than those given by Theorem 1.5 can be obtained.

The next theorem, which is taken from [3], gives sufficient conditions for the continuity of $\{(\times_{j=1}^n L_j)(\mu_x, t), (x, t) \in (R^d)^n \times (R_+)^n\}$. It is particularly interesting when μ is radially symmetric on $(R^d)^n$.

THEOREM 1.6 (Continuity of CAFs). *Let $\{u_j^1\}_{j=1}^n$ be the 1-potentials of strongly symmetric Lévy processes on R^d. Let $h_j: R^+ \to R^+$, $1 \le j \le n$ be increasing and be such that*

$$\int_{R^d} \frac{\widehat{u_j^2}(\zeta)}{h_j(|\zeta|)}\, d\zeta < \infty \qquad \forall\, 1 \le j \le n. \tag{1.45}$$

Let μ be a finite positive measure on $(R^d)^n$. If

$$|\hat{\mu}(\xi)| = O\biggl(\frac{1}{\prod_{j=1}^{n} h_j(|\xi|)(\log|\xi|)^{2n+\delta}}\biggr)^{1/2}, \tag{1.46}$$

$\xi \in (R^d)^n$, for some $\delta > 0$, then $\{(\times_{j=1}^{n} L_j)(\mu_x, t), (x,t) \in (R^d)^n \times (R_+)^n\}$ is continuous almost surely on $(R^d)^n$.

In order to compare the conditions in Theorems 1.4 and 1.6 we give examples when (1.46) is satisfied with $u_1^1 = \cdots = u_n^1 =_{\text{def}} u^1$. The following are taken from [3].

EXAMPLES 1.2. Each of the following conditions on u^1 and $\hat{\mu}$ imply that $\{(\times_{j=1}^{n} L_j)(\mu_x, t), (x,t) \in (R^d)^n \times (R_+)^n\}$ is continuous almost surely, where $d = 1, 2$ or 3 according to whether $\xi \in (R^1)^n$, $(R^2)^n$ or $(R^3)^n$.

(i) u^1 is the 1-potential of Brownian motion in R^2 and

$$|\hat{\mu}(\xi)| = O\Bigl(\log^{-(2n+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.47}$$

$\xi \in (R^2)^n$, for some $\varepsilon > 0$.

(ii) u^1 is the 1-potential of Brownian motion in R^3 and

$$|\hat{\mu}(\xi)| = O\Bigl(|\xi|^{-n} \log^{-(3n/2+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.48}$$

$\xi \in (R^3)^n$, for some $\varepsilon > 0$.

(iii) u^1 is the 1-potential of a symmetric stable process in R^2 of index $1 < \alpha < 2$ and

$$|\hat{\mu}(\xi)| = O\Bigl(|\xi|^{-(2-\alpha)n} \log^{-(3n/2+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.49}$$

$\xi \in (R^2)^n$, for some $\varepsilon > 0$.

(iv) u^1 is the 1-potential of a symmetric stable process in R^3 of index $3/2 < \alpha < 2$ and

$$|\hat{\mu}(\xi)| = O\Bigl(|\xi|^{-(3-\alpha)n} \log^{-(3n/2+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.50}$$

$\xi \in (R^3)^n$, for some $\varepsilon > 0$,

(v) u^1 is the 1-potential of a symmetric stable process in R^1 of index 1 and

$$|\hat{\mu}(\xi)| = O\Bigl(\log^{-(2n+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.51}$$

$\xi \in (R^1)^n$, for some $\varepsilon > 0$.

(vi) u^1 is the 1-potential of a symmetric stable process in R^1 of index $1/2 < \alpha < 1$ and

$$|\hat{\mu}(\xi)| = O\Bigl(|\xi|^{-(1-\alpha)n} \log^{-(3n/2+\varepsilon)}(|\xi|)\Bigr) \quad \text{as } |\xi| \to \infty, \tag{1.52}$$

$\xi \in (R^1)^n$, for some $\varepsilon > 0$.

Clearly, given (1.46), it is easy to obtain examples when $\{(\times_{j=1}^{n} L_j)(\mu_x, t), (x, t) \in (R^d)^n \times (R_+)^n\}$ is continuous almost surely on $(R^d)^n$ and the 1-potentials are not all equal.

The rest of this paper is organized as follows. In Section 2 we develop the isomorphism theorem which will be used in the study of CAFs of independent Lévy processes. In Section 3 we use this isomorphism theorem to prove Theorem 1.2. In Section 4 we develop a different isomorphism theorem which in Section 5 we use to prove Theorem 1.3. In the final section we explain how the concrete sufficient continuity conditions in Theorems 1.4–1.6 are obtained from the results of [3].

2. Isomorphism theorem for CAFs of several independent Lévy processes. In this section we extend a version of an isomorphism theorem of Dynkin to obtain a relationship between CAFs of several independent Lévy processes and Gaussian chaos processes. For $\nu \in \mathscr{G}_j^2$ we define the second-order Gaussian chaos

$$H_j(\nu) = \lim_{\delta \to 0} \int H_{j,x,\delta}\, d\nu(x), \tag{2.1}$$

where $H_{j,x,\delta}$ is given in (1.22). It is easy to check that $H(\mu)$ has mean 0 and that for $\mu, \nu \in \mathscr{G}_j^2$,

$$EH_j(\mu)H_j(\nu) = \int\int \left(u_j^1(x-y)\right)^2 d\mu(x)\, d\nu(y). \tag{2.2}$$

When $\mathscr{G}_j^2 \neq \varnothing$ one can show that $\rho_{\varepsilon,y}(dx) \in \mathscr{G}_j^2$. Let

$$H_j(y,\varepsilon) =_{\text{def}} H_j(\rho_{\varepsilon,y}). \tag{2.3}$$

When $\mu \in \mathscr{G}_n^{2,n}$ we define

$$\left(\mathop{\times}_{j=1}^{n} H_j\right)(\varepsilon,\mu) = \int \prod_{j=1}^{n} H_j(x_j,\varepsilon)\, d\mu(x_1,\dots,x_n) \tag{2.4}$$

for $\varepsilon > 0$. One can check that

$$\left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu) =_{\text{def}} \lim_{\varepsilon \to 0}\left(\mathop{\times}_{j=1}^{n} H_j\right)(\varepsilon,\mu) \tag{2.5}$$

exists as a limit in L^2 and satisfies

$$E\left\{\left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu)\right\} = 0 \tag{2.6}$$

and

$$\begin{aligned} &E\left\{\left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu)\left(\mathop{\times}_{j=1}^{n} H_j\right)(\nu)\right\} \\ &\quad = 2^n \int\int \prod_{j=1}^{n} \left(u_j^1(x_j-y_j)\right)^2 d\mu(x_1,\dots,x_n)\, d\nu(y_1,\dots,y_n) \end{aligned} \tag{2.7}$$

for all $\mu, \nu \in \mathscr{G}_n^{2,n}$. We also note that for $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ and $\mu \in \mathscr{G}_n^{2,n}$,

$$\begin{aligned} & E_{G_1,\dots,G_n}\left(\left\{\left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu_a) - \left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu_b)\right\}^2\right) \\ & \quad = 2^n \int\int \prod_{j=1}^{n} \left(u_j^1(x_j - y_j)\right)^2 \left(d\mu_a(x_1,\dots,x_n) - d\mu_b(x_1,\dots,x_n)\right) \\ & \qquad\qquad \left(d\mu_a(y_1,\dots,y_n) - d\mu_b(y_1,\dots,y_n)\right) \\ & \quad = 2^{n+1} \int\int \left(\prod_{j=1}^{n} \left(u_j^1(x_j - y_j)\right)^2 - \prod_{j=1}^{n} \left(u_j^1(x_j + a_j - y_j - b_j)\right)^2\right) \\ & \qquad\qquad d\mu(x_1,\dots,x_n)\, d\mu(y_1,\dots,y_n). \end{aligned} \tag{2.8}$$

The following version of an isomorphism theorem of Dynkin is Theorem 2.2 in [6], adapted to the needs of this paper. It relates continuous additive functionals of a single strongly symmetric Lévy processes X with 1-potential density u^1 to second-order Gaussian chaos processes $H(\mu)$, defined as in (2.1), with u^1 in place of u_j^1. We use L_t^{μ} to denote the continuous additive functional of X with Revuz measure μ and $\text{Rev}(X)$ to denote the class of Revuz measures of X. Let λ be a mean 1 exponential random variable independent of everything else.

THEOREM 2.7. *Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of finite positive measures in $\mathscr{G}^2 \cap \text{Rev}(X)$. Set $L^{\mu_\cdot} = (L_\lambda^{\mu_1}, L_\lambda^{\mu_2}, \dots)$ and $H(\mu_\cdot) = (H(\mu_1), H(\mu_2), \dots)$. Then for any finite measure $\rho \in \mathscr{G}^1$, function g with $g \cdot dx \in \mathscr{G}^1$ and $\mathscr{C}$ measurable nonnegative function F on R^∞,*

$$E_G E_\lambda^\rho\left(F\left(L^{\mu_\cdot} + \tfrac{1}{2}H(\mu_\cdot)\right)g(X_\lambda)\right) = E_G\left(F\left(\tfrac{1}{2}H(\mu_\cdot)\right)G_\rho G_{g\cdot dx}\right), \tag{2.9}$$

where $g \cdot dx$ denotes the measure in R^d with density g and $\mathscr{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

Actually, Theorem 2.2 in [6] is stated for compactly supported $\rho \in \mathscr{G}^1$, but the extension to any finite measure $\rho \in \mathscr{G}^1$ is immediate.

For $x \in R^d$ and $t \in R_+$ let

$$L_{j,t}^{x,\delta} =_{\text{def}} \int_0^t f_\delta\left(X_j(s) - x\right) ds, \tag{2.10}$$

where f_δ is defined just after (1.18). Let λ_j be independent mean 1 exponential random variables, and let $\rho_j \in \mathscr{G}_j^1$ be compactly supported probability measures. We now define for each subset $A \subseteq \{1, \dots, n\}$, $\varepsilon > 0$ and $\mu \in \mathscr{G}_n^{2,n}$,

$$\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon, \mu) = \int \prod_{i\in A} L_{i,\lambda_i}^{x_i,\varepsilon} \prod_{j\in A^c} H_j(x_j, \varepsilon)\, d\mu(x_1, \dots, x_n). \tag{2.11}$$

In the course of proving the next theorem we show that for each subset $A \subseteq \{1, \dots, n\}$, $(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j)(\varepsilon, \mu)$ converges, as $\varepsilon \to 0$, in the L^2 space for the measure

$$E_{\bar G} E_{\bar\lambda}^{\bar\rho} =_{\text{def}} \prod_{j=1}^n E_{G_j} \times E_{\lambda_j}^{\rho_j} \tag{2.12}$$

for all $\mu \in \mathscr{G}_n^{2,n}$. (Here E_{G_j} denotes expectation with respect to the probability space of $\{G_{j,\nu}, \nu \in \mathscr{G}_j^1\}$ defined just before (1.18)).

We set

$$\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\mu) =_{\text{def}} \lim_{\varepsilon\to 0}\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon, \mu). \tag{2.13}$$

To unify the notation we sometimes write $(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j)(0, \mu)$ for $(\bigtimes_{i\in A} \cdot L_i \bigtimes_{j\in A^c} H_j)(\mu)$.

The following isomorphism theorem is the main ingredient in the proof of Theorem 1.2.

THEOREM 2.8. *Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers and $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite measures in $\mathscr{G}_n^{2,n}$. For any $A \subseteq \{1, \dots, n\}$ set*

$$\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon_\cdot, \mu_\cdot) = \Big\{\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon_1, \mu_1), \Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon_2, \mu_2), \dots\Big\}. \tag{2.14}$$

Then, for any $B \subseteq \{1, \dots, n\}$, finite measures $\rho_j \in \mathscr{G}_j^1$, $j \in B$, functions g_j with $g_j \cdot dx \in \mathscr{G}_j^1$, $j \in B$, and $\mathscr{C}$ measurable nonnegative function F on R^∞,

$$\begin{aligned} &E_{\bar G} E_{\bar\lambda}^{\bar\rho}\bigg(F\bigg(\sum_{A\subseteq B} \frac{1}{2^{|A^c|}}\Big(\bigtimes_{i\in A} L_i \bigtimes_{j\in A^c} H_j\Big)(\varepsilon_\cdot, \mu_\cdot)\bigg) \prod_{j\in B} g_j(X_{j,\lambda_j})\bigg) \\ &\quad = E_{\bar G}\bigg(F\bigg(\frac{1}{2^n}\Big(\bigtimes_{j=1}^n H_j\Big)(\varepsilon_\cdot, \mu_\cdot)\bigg) \prod_{j\in B} G_{j,\rho_j} G_{j, g_j\cdot dx}\bigg), \end{aligned} \tag{2.15}$$

where $\mathscr{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

PROOF. Consider first the case when all $\varepsilon_k > 0$, all μ_k are finite sums of point masses and F is a bounded continuous function which depends only on a finite number of arguments. In this case the theorem follows from Theorem 2.7 using the equation

$$\begin{aligned}&\prod_{i\in B}\Big(L^{x_i,\varepsilon}_{i,\lambda_i}+\tfrac{1}{2}H_i(x_i,\varepsilon)\Big)\prod_{i\in B^c}\Big(\tfrac{1}{2}H_i(x_i,\varepsilon)\Big)\\ &\quad=\sum_{A\subseteq B}\frac{1}{2^{|B-A|}}\prod_{i\in A}L^{x_i,\varepsilon}_{i,\lambda_i}\prod_{j\in B-A}H_j(x_j,\varepsilon)\prod_{i\in B^c}\Big(\tfrac{1}{2}H_i(x_i,\varepsilon)\Big)\end{aligned}\tag{2.16}$$

and taking expectations in (2.15), with respect to $E_{G_j}\times E^{\rho_j}_{\lambda_j}$, $j=1,\dots,n$, one at a time.

We next use the fact that the integrand in (2.11) is continuous in the variables $x_1,\dots,x_n$ in L^2 of the probability space. This implies that it is the limit in L^2, and hence almost surely, of a sequence of similar integrals in which μ is replaced by a finite sum of point masses. Thus we can remove the restriction on the μ_k. We can then, in turn, remove the restriction on F.

We now have established (2.15) in the case when all the $\varepsilon_k>0$. We use this result to show, inductively on $|A|$, that the limits in (2.13) exist in L^2. Restricting F to be a bounded continuous function which depends only on a finite number of arguments, we can now take the limit as the $\varepsilon_k\to 0$ in (2.15). We can then lift the restriction on F, completing the proof of Theorem 2.8. □

In Theorem 2.8 the requirement, that $g_j\cdot dx\in\mathscr{G}^1_j$, is not satisfied when $g_j\equiv 1$. Therefore we must take into account the value of X_λ. This is an annoying condition which we can remove when the 1-potentials u^1_j of the independent Lévy processes X_j, $j=1,\dots,n$, all satisfy (1.1). Let $\|\cdot\|$ be a norm on ℓ^∞. (For a function in ℓ^∞, say $\{h(\varepsilon_k,\mu)\}_{k=1}^\infty$, we write $\|h\|=\|h(\varepsilon,\mu)\|$ to keep track of the variables being considered).

In the next theorem we obtain a useful corollary of Theorem 2.8.

THEOREM 2.9. *Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers and let $\{\mu_k\}_{k=1}^\infty$ be a sequence of finite measures in $\mathscr{G}^{2,n}_n$. There exists a constant p' depending only on u^1_j, $j=1,\dots,n$ and a constant C depending only on n, ρ_j and u^1_j, $j=1,\dots,n$ and p', such that*

$$E^{\bar\rho}_{\bar\lambda}\left\|\Big(\mathop{\times}_{i=1}^n L_i\Big)(\varepsilon,\mu)\right\|\le C\left(E_{\bar G}\left\|\Big(\mathop{\times}_{j=1}^n H_j\Big)(\varepsilon,\mu)\right\|^{p'}\right)^{1/p'}.\tag{2.17}$$

PROOF. Let

$$\mathbf{W}=_{\mathrm{def}}\sum_{A\subseteq[0,\dots,n]}\frac{1}{2^{|A^c|}}\Big(\mathop{\times}_{i\in A}L_i\mathop{\times}_{j\in A^c}H_j\Big)(\varepsilon,\mu).\tag{2.18}$$

By the convexity of the norm and the fact that the independent second-order chaoses $H_j(x_j, \varepsilon)$ in (2.11) all have mean zero, we have that

$$
\begin{aligned}
E_{\bar G}\|\mathbf{W}\| &\ge \|E_{\bar G}\mathbf{W}\| \\
&= \left\| \left(\mathop{\times}_{i=1}^{n} L_i \right)(\varepsilon, \mu) \right\|.
\end{aligned}
\tag{2.19}
$$

Consequently,

$$
\begin{aligned}
E_{\bar\lambda}^{\bar\rho}\left(\left\| \left(\mathop{\times}_{i=1}^{n} L_i \right)(\varepsilon, \mu) \right\| \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right) &\le E_{\bar\lambda}^{\bar\rho}\left(\|E_{\bar G}\mathbf{W}\| \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right) \\
&\le E_{\bar G} E_{\bar\lambda}^{\bar\rho}\left(\|\mathbf{W}\| \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right).
\end{aligned}
\tag{2.20}
$$

By the Hölder inequality,

$$
\begin{aligned}
&E_{\bar\lambda}^{\bar\rho}\left(\|\mathbf{W}\| \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right) \\
&\quad \le \left(E_{\bar\lambda}^{\bar\rho}\left(\|\mathbf{W}\|^p \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right) \right)^{1/p} \left(E_{\bar\lambda}^{\bar\rho} \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right)^{1/q},
\end{aligned}
\tag{2.21}
$$

where $1/p + 1/q = 1$, $p, q > 1$. By Theorem 2.8, the Schwarz inequality applied twice and the fact that for a mean zero normal random variable g, $Eg^4 = 3(Eg^2)^2$,

$$
\begin{aligned}
&E_{\bar\lambda}^{\bar\rho}\left(\|\mathbf{W}\|^p \prod_{j=1}^{n} g_j(X_{j,\lambda_j}) \right) \\
&\quad \le \left(E_{\bar G} \left\| \frac{1}{2^n} \left(\mathop{\times}_{j=1}^{n} H_j \right)(\varepsilon, \mu) \right\|^{2p} \right)^{1/2} \left(3^n \prod_{j=1}^{n} EG_{j,\rho_j}^2 EG_{j,g_j\cdot dx}^2 \right)^{1/4}.
\end{aligned}
\tag{2.22}
$$

Let $p' = 2p$.

For each $j = 1, \dots, n$ let $\{\Delta_{k_j}\}_{k_j=1}^{\infty}$ be a partition of R^d into cubes of volume one. Let $g_{j,k_j} =_{\mathrm{def}} I_{[\Delta_{k_j}]}$. Then

$$
\begin{aligned}
EG_{j,g_{j,k_j}\cdot dx}^2 &= \int\int u^1(x-y) g_{j,k_j}(x) g_{j,k_j}(x)\, dx\, dy \\
&= \int_{[0,1]^d}\int_{[0,1]^d} u^1(x-y)\, dx\, dy
\end{aligned}
\tag{2.23}
$$

$$
=_{\mathrm{def}} \mathbf{U}. \tag{2.24}
$$

Let

$$
\mathbf{V} =_{\mathrm{def}} \left(E_{\bar G} \left\| \left(\mathop{\times}_{j=1}^{n} H_j \right)(\varepsilon, \mu) \right\|^{p'} \right)^{1/p'}. \tag{2.25}
$$

Using (2.20)–(2.25) we see that

$$
\begin{aligned}
&E_{\bar\lambda}^{\bar\rho}\left(\left\|\left(\mathop{\times}_{i=1}^{n} L_i\right)(\varepsilon,\mu)\right\|\right)\\
&\quad=\sum_{k_1,\dots,k_n} E_{\bar\lambda}^{\bar\rho}\left(\left\|\left(\mathop{\times}_{i=1}^{n} L_i\right)(\varepsilon,\mu)\right\|\prod_{j=1}^{n} g_{j,k_j}(X_{j,\lambda_j})\right)\\
&\quad\le\sum_{k_1,\dots,k_n}\left(E_{\bar\lambda}^{\bar\rho}\left(\|\mathbf{W}\|^p\prod_{j=1}^{n} g_{j,k_j}(X_{j,\lambda_j})\right)\right)^{1/p}\left(E_{\bar\lambda}^{\bar\rho}\prod_{j=1}^{n} g_{j,k_j}(X_{j,\lambda_j})\right)^{1/q}\\
&\quad\le C\mathbf{VU}^{n/(4p)}\prod_{j=1}^{n}\left(EG_{j,\rho_j}^2\right)^{1/(4p)}\sum_{k_1,\dots,k_n}\left(E_{\bar\lambda}^{\bar\rho}\prod_{j=1}^{n} g_{j,k_j}(X_{j,\lambda_j})\right)^{1/q}.
\end{aligned}
\tag{2.26}
$$

Thus to obtain (2.15) we need only show that this last sum is finite. Note that

$$
\sum_{k_1,\dots,k_n}\left(E_{\bar\lambda}^{\bar\rho}\prod_{j=1}^{n} g_{j,k_j}(X_{j,\lambda_j})\right)^{1/q}=\prod_{j=1}^{n}\left(\sum_{k_j}\left(P^{\rho_j}(X_{j,\lambda_j}\in\Delta_{k_j})\right)^{1/q}\right)
\tag{2.27}
$$

and since u_j^1 is the probability density of X_{j,λ_j} we see that

$$
\begin{aligned}
\sum_{k_j}\left(P^{\rho_j}(X_{j,\lambda_j}\in\Delta_{k_j})\right)^{1/q}&=\sum_{k_j}\left(\int\int_{\Delta_{k_j}} u_j^1(x-y)\,dx\,\rho_j(dy)\right)^{1/q}\\
&\le\int\sum_{k_j}\int_{\Delta_{k_j}}\left(u_j^1(x-y)\right)^{1/q}dx\,\rho_j(dy)\\
&=\int\left(u_j^1(x)\right)^{1/q}dx.
\end{aligned}
\tag{2.28}
$$

Using Hölder's inequality, it is easy to show that, for q sufficiently close to 1, (1.1) implies that this last integral is finite.

When $\mu\in\mathscr{G}_n^{2,n}$ it can be seen from (2.8) using Fourier transforms (see Lemma 3.1 of [4]) that $\{(\times_{j=1}^n H_j)(\mu_x);\, x\in(R^d)^n\}$ is continuous in $L^2_{\tilde G}$. Set

$$
\left(\mathop{\times}_{j=1}^{n} H_j\right)_{\varepsilon}(\mu_x)=_{\mathrm{def}}\int\left(\mathop{\times}_{j=1}^{n} H_j\right)(\mu_y)\prod_{j=1}^{n} f_{\varepsilon}(x_j-y_j)\,dy_j
\tag{2.29}
$$

and recall the definition of $(\times_{j=1}^n H_j)(\varepsilon,\mu_x)$ given in (2.4). In applying Theorem 2.9 in the proof of Theorem 1.2 we use the following relationship.

LEMMA 2.1.

$$
\left(\mathop{\times}_{j=1}^{n} H_j\right)_{\varepsilon}(\mu_x)=\left(\mathop{\times}_{j=1}^{n} H_j\right)(\varepsilon,\mu_x)
\tag{2.30}
$$

in $L^2_{\tilde G}$ *for each* $x\in(R^d)^n$ *and* $\varepsilon>0$.

PROOF. All the limits in this proof are taken in $L^2_{\tilde{G}}$. Using the relevant definitions, we have

$$\begin{aligned}
&\Big(\mathop{\times}_{j=1}^{n} H_j\Big)_{\varepsilon}(\mu_x)\\
&\quad= \int \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\mu_y) \prod_{j=1}^{n} f_{\varepsilon}(x_j - y_j)\, dy_j \qquad (2.31)\\
&\quad= \int \lim_{\delta\to 0} \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\delta, \mu_y) \prod_{j=1}^{n} f_{\varepsilon}(x_j - y_j)\, dy_j.
\end{aligned}$$

Using Fourier transforms as in Lemma 3.1 in [4], we see that the convergence

$$\lim_{\delta\to 0} \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\delta, \mu_y) = \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\mu_y) \qquad (2.32)$$

is uniform in $y \in (R^d)^n$ so that the last display in (2.31) is equal to

$$\begin{aligned}
&\lim_{\delta\to 0} \int \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\delta, \mu_y) \prod_{j=1}^{n} f_{\varepsilon}(x_j - y_j)\, dy_j\\
&\quad= \lim_{\delta\to 0} \int \bigg(\int \prod_{j=1}^{n} H_j(z_j, \delta)\, d\mu_y(z_1, \dots, z_n)\bigg) \prod_{j=1}^{n} f_{\varepsilon}(x_j - y_j)\, dy_j\\
&\quad= \lim_{\delta\to 0} \int \bigg(\int \prod_{j=1}^{n} H_j(z_j, \delta)\, d\mu_{y+x}(z_1, \dots, z_n)\bigg) \prod_{j=1}^{n} f_{\varepsilon}(y_j)\, dy_j\\
&\quad= \lim_{\delta\to 0} \int \bigg(\int \prod_{j=1}^{n} H_j(z_j - y_j, \delta)\, d\mu_x(z_1, \dots, z_n)\bigg) \prod_{j=1}^{n} f_{\varepsilon}(y_j)\, dy_j \qquad (2.33)\\
&\quad= \lim_{\delta\to 0} \int \bigg(\int \prod_{j=1}^{n} H_j(z_j - y_j, \delta) \prod_{j=1}^{n} f_{\varepsilon}(y_j)\, dy_j\bigg)\, d\mu_x(z_1, \dots, z_n)\\
&\quad= \lim_{\delta\to 0} \int \prod_{j=1}^{n} H_j(f_{\varepsilon, z_j} * f_{\delta})\, d\mu_x(z_1, \dots, z_n)\\
&\quad= \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\varepsilon, \mu_x),
\end{aligned}$$

where we used the fact that

$$\int H_j(z_j - y_j, \delta) f_{\varepsilon}(y_j)\, dy_j = H_j(f_{\varepsilon, z_j} * f_{\delta}) \qquad (2.34)$$

and that the convergence in

$$\lim_{\delta\to 0} H_j(f_{\varepsilon, z_j} * f_{\delta}) = H_j(f_{\varepsilon, z_j}) \equiv H_j(z_j, \varepsilon) \qquad (2.35)$$

is uniform in $z_j \in R^d$ by the same reasoning as above.

The next lemma, which will also be used in the proof of Theorem 1.2, enables us to show that two different representations of Gaussian chaoses are equivalent.

LEMMA 2.2. *For the processes defined in* (1.25) *and* (2.4) *we have*

$$\mathscr{H}_2(y;\mu) = \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\mu_y) \tag{2.36}$$

in $L^2_{\tilde{G}}$ *for each* $y \in (R^d)^n$.

PROOF. All the limits in this proof are taken in $L^2_{\tilde{G}}$. Let $\rho_{\varepsilon,x}(ds) =_{\text{def}} \prod_{j=1}^n \rho_{\varepsilon,x_j}(ds_j)$. Using the same arguments as in the proof of Lemma 2.1 we have

$$\begin{aligned}
\mathscr{H}_2(y;\mu) &= \lim_{\varepsilon\to 0}\int \mathscr{H}_2(s+y;\mu)\,\rho_{\varepsilon,0}(ds)\\
&= \lim_{\varepsilon\to 0}\int\Big(\int \lim_{\delta\to 0}\prod_{j=1}^n H_{j,s_j+x_j,\delta}\,d\mu_y(x_1,\dots,x_n)\Big)\rho_{\varepsilon,0}(ds)\\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int\Big(\int\prod_{j=1}^n H_{j,s_j,\delta}\,\rho_{\varepsilon,x_j}(ds_j)\Big)\,d\mu_y(x_1,\dots,x_n)\\
&= \lim_{\varepsilon\to 0}\int\prod_{j=1}^n \lim_{\delta\to 0}\Big(\int H_{j,s_j,\delta}\,\rho_{\varepsilon,x_j}(ds_j)\Big)\,d\mu_y(x_1,\dots,x_n)\\
&= \lim_{\varepsilon\to 0}\int\prod_{j=1}^n H(x_j,\varepsilon)\,d\mu_y(x_1,\dots,x_n)\\
&= \Big(\mathop{\times}_{j=1}^{n} H_j\Big)(\mu_y).
\end{aligned} \tag{2.37}$$

3. Continuity theorem for CAFs of several independent Lévy processes.

PROOF OF THEOREM 1.2. Let $Y_j(t)$, $j = 1,\dots,n$, denote the Lévy processes $X_j(t)$, $j = 1,\dots,n$, killed at independent mean-1 exponential times λ_j. We prove Theorem 1.2 for $Y_j(t)$, $j = 1,\dots,n$. The stated result then follows by Fubini's theorem. The advantage of this approach is that it allows us to suppress the λ_j in Theorems 2.8 and 2.9. [In (2.11) we replace λ_i by ∞.] To avoid confusing notation we still use the 1-potential densities u^1_j of the unkilled processes X_j, which are the 0-potential densities of the killed processes Y_j. It is clear that Theorems 2.8 and 2.9 remain valid with these changes.

Let $t = (t_1,\dots,t_n)$. For each $\varepsilon > 0$, set

$$\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\varepsilon,\mu_x,t) =_{\text{def}} \int\Big\{\prod_{j=1}^n\int_0^{t_j} f_\varepsilon(Y_j(s_j)-y_j)\,ds_j\Big\}\,d\mu_x(y). \tag{3.1}$$

Note that

$$\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\varepsilon,\mu_x,\infty,\dots,\infty) = \Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\varepsilon,\mu_x), \tag{3.2}$$

which is defined in (2.10) and (2.11).

In proving this theorem we show first that $\{(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t),(x,t)\in (R^d)^n\times(R_+)^n\}$ converges almost surely, locally uniformly as $\varepsilon\to 0$. After showing this we then identify the almost sure continuous limit with $\{(\times_{j=1}^n L_j)(\mu_x,t),(x,t)\in(R^d)^n\times(R_+)^n\}$.

We proceed with the first step. For each $A\subseteq\{1,\dots,n\}$, $(\varepsilon,t)\in(0,1]\times(R_+)^n$ and $x,v,y\in(R^d)^n$, let

$$
\begin{aligned}
&M_A(\varepsilon,\mu_x,t_A,v_{A^c})\\
&\quad=_{\mathrm{def}} E_A^{y_A}\left(E_{A^c}^{v_{A^c}}\left\{\left(\mathop{\times}_{j=1}^{n}L_j\right)(\varepsilon,\mu_x,\infty,\dots,\infty)\right\}\Bigg|\bigotimes_{j\in A}\mathcal{F}_{j,t_j}\right).
\end{aligned}
\tag{3.3}
$$

Here, E_A denotes expectation with respect to the Markov processes $\{Y_j;\ j\in A\}$, $t_A=\{t_j;\ j\in A\}$ and $v_{A^c}=\{v_j;\ j\in A^c\}$. $M_A(\varepsilon,\mu_x,t_A,v_{A^c})$ is a martingale in each $\{t_j;\ j\in A\}$ individually. We see below, as indicated by the notation, that $M_A(\varepsilon,\mu_x,t_A,v_{A^c})$ does not depend on y. Note that when $A=\varnothing$, $E_A^{y_A}$ is taken to be the identity.

We say that a sequence of functions $Z_\varepsilon(x)$ converges rationally locally uniformly on some open subset V of a Euclidean space W, as $\varepsilon\to 0$, if for each compact $K\subseteq V$, $Z_\varepsilon(x)$ converges uniformly on the rational elements of K, as $\varepsilon\to 0$ through rational values.

The next lemma is the key step in the proof of Theorem 1.2. This is where the isomorphism theorem is used.

LEMMA 3.1. *Under the hypotheses of Theorem* 1.2, *for each* $A\subseteq\{1,\dots,n\}$, *including* $A=\varnothing$, $M_A(\varepsilon,\mu_x,t_A,v_{A^c})$ *converges almost surely rationally locally uniformly in* $(x,v,t)\in(R^d)^n\times(R^d)^n\times(R_+)^n$ *as* $\varepsilon\to 0$.

Before giving the proof of Lemma 3.1 we show that it implies that $(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t)$ converges almost surely locally uniformly in $(x,t)\in(R^d)^n\times(R_+)^n$ as $\varepsilon\to 0$. This is a consequence of the following lemma which actually shows more.

LEMMA 3.2. *Under the hypotheses of Theorem* 1.2, *for each* $A\subseteq\{1,\dots,n\}$, *including* $A=\varnothing$, $E_{A^c}^{v_{A^c}}\{(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t_A,\infty_{A^c})\}$ *converges almost surely locally uniformly in* $(x,v,t)\in(R^d)^n\times(R^d)^n\times(R_+)^n$ *as* $\varepsilon\to 0$.

The notation $(t_A,\infty_{A^c})=_{\mathrm{def}}(s_1,\dots,s_n)$ where $s_j=t_j$ whenever $j\in A$ and $s_j=\infty$ whenever $j\in A^c$. Note that when $A=\{1,\dots,n\}$, $E_{A^c}^{v_{A^c}}\{(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t_A,\infty_{A^c})\}=(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t)$. Thus Lemma 3.2 does indeed show that $(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t)$ converges almost surely locally uniformly in $(x,t)\in(R^d)^n\times(R_+)^n$ as $\varepsilon\to 0$.

PROOF. Since $E_{A^c}^{v_{A^c}}\{(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t_A,\infty_{A^c})\}$ is clearly continuous in $(\varepsilon, x, v, t)\in(0,1]\times(R^d)^n\times(R^d)^n\times(R_+)^n$, it suffices to show that $E_{A^c}^{v_{A^c}}\{(\times_{j=1}^n L_j)(\varepsilon,\mu_x,t_A,\infty_{A^c})\}$ converges almost surely rationally locally uniformly in $(x, v,$

$t) \in (R^d)^n \times (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$. We do this by induction on $|A|$. When $A = \emptyset$,

$$E^{v_{A^c}}_{A^c}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_A, \infty_{A^c})\right\} = E^{v}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t, \infty, \dots, \infty)\right\} \tag{3.4}$$
$$= M_\emptyset(\varepsilon, \mu_x, t_\emptyset, v).$$

Therefore, in this case, the assertion of this lemma is given by Lemma 3.1.

We now describe how the induction argument is carried out. Assume that the statement in this lemma is true for all $A \subseteq \{1, \dots, n\}$ with $|A| < k$, and choose some $A \subseteq \{1, \dots, n\}$ with $|A| = k$. Using additivity and the Markov property, we see that for any $y = (y_1, \dots, y_n) \in (R^d)^n$,

$$\begin{aligned}
&M_A(\varepsilon, \mu_x, t_A, v_{A^c}) \\
&\quad = E^{y_A}_{A}\left(E^{v_{A^c}}_{A^c}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, \infty, \dots, \infty)\right\}\Big| \bigotimes_{j\in A} \mathscr{F}_{j,t_j}\right) \\
&\quad = E^{y_A, v_{A^c}}_{A, A^c}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, \infty, \dots, \infty)\Big| \bigotimes_{j\in A} \mathscr{F}_{j,t_j}\right\} \\
&\quad = E^{y_A, v_{A^c}}_{A, A^c}\left\{\sum_{B\subseteq A}\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_B, \infty_{B^c}) \circ \prod_{j\in A-B} \theta_{j,t_j}\Big| \bigotimes_{j\in A} \mathscr{F}_{j,t_j}\right\} \\
&\quad = \sum_{B\subseteq A} E^{v_{A^c}, Y_{A-B}(t_{A-B})}_{A^c, A-B}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_B, \infty_{B^c})\right\},
\end{aligned} \tag{3.5}$$

where $Y_{A-B}(t_{A-B}) = \{Y_j(t_j);\ j \in A - B\}$. Note that when $B = A$,

$$\begin{aligned}
&E^{v_{A^c}, Y_{A-B}(t_{A-B})}_{A^c, A-B}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_B, \infty_{B^c})\right\} \\
&\quad = E^{v_{A^c}}_{A^c}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_A, \infty_{A^c})\right\}.
\end{aligned} \tag{3.6}$$

All other terms in the last equality of (3.5) are of the form

$$E^{z_{B^c}}_{B^c}\left\{\left(\mathop{\times}_{j=1}^{n} L_j\right)(\varepsilon, \mu_x, t_B, \infty_{B^c})\right\} \tag{3.7}$$

with $|B| < |A| = k$. By the induction hypothesis these terms converge almost surely locally uniformly in $(x, z, t) \in (R^d)^n \times (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$. Furthermore, by Lemma 3.1, $M_A(\varepsilon, \mu_x, t_A, v_{A^c})$ converges almost surely locally uniformly in $(x, v, t) \in (R^d)^n \times (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$ and hence so does the sum in the last line of (3.5). This shows that the last line in (3.6) converges almost surely locally uniformly in $(x, v, t) \in (R^d)^n \times (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$, which is what we set out to prove. This completes the proof of Lemma 3.2. □

To complete the proof that $(\times_{j=1}^{n} L_j)(\varepsilon, \mu_x, t)$ converges almost surely locally uniformly in $(x, t) \in (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$ it only remains to prove Lemma 3.1. This follows from the isomorphism theorem, Theorem 2.9.

PROOF OF LEMMA 3.1. Note that by Lemma 2.2, $\{\mathscr{H}_2(x;\mu),\ x\in(R^d)^n\}$ and $\{(\times_{j=1}^n H_j)(\mu_x),\ x\in(R^d)^n\}$ are equivalent stochastic processes. Consequently, by hypothesis, $\{(\times_{j=1}^n H_j)(\mu_x),\ x\in(R^d)^n\}$ is continuous almost surely. This implies that $(\times_{j=1}^n H_j)_\varepsilon(\mu_x)$, defined in (2.29), is continuous in $x\in(R^d)^n$ almost surely and that it converges locally uniformly to $(\times_{j=1}^n H_j)(\mu_x)$ almost surely as $\varepsilon\to 0$.

As a consequence of this, we see that for any $\gamma>0$, we can find a $\delta>0$, such that

$$E_{\bar G}\left(\sup_{\substack{x\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\left(\mathop{\times}_{j=1}^n H_j\right)_\varepsilon(\mu_x)-\left(\mathop{\times}_{j=1}^n H_j\right)_{\varepsilon'}(\mu_x)\right|\right)\le\gamma, \tag{3.8}$$

where $B(m)$ denotes the ball of radius m in $(R^d)^n$. Furthermore, since all moments of norms of Gaussian chaoses are equivalent, (see, example, (4.1) in [1]), we also have that for all $p>0$ there exists a constant C_p, depending only on p, such that

$$E_{\bar G}\left(\sup_{\substack{x\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\left(\mathop{\times}_{j=1}^n H_j\right)_\varepsilon(\mu_x)-\left(\mathop{\times}_{j=1}^n H_j\right)_{\varepsilon'}(\mu_x)\right|^p\right)\le C_p\gamma^p. \tag{3.9}$$

By (2.30), $(\times_{j=1}^n H_j)_\varepsilon(\mu_x)=(\times_{j=1}^n H_j)(\varepsilon,\mu_x)$ in $L^2_{\bar G}$ for each $\varepsilon>0$ and $x\in(R^d)^n$. Therefore, we have that for any countable dense subset C_δ of $(R^d)^n\times(0,\delta]^2$,

$$E_{\bar G}\left(\sup_{(x,\,\varepsilon,\,\varepsilon')\in C_\delta\cap B(m)}\left|\left(\mathop{\times}_{j=1}^n H_j\right)(\varepsilon,\mu_x)-\left(\mathop{\times}_{j=1}^n H_j\right)(\varepsilon',\mu_x)\right|^p\right)\le C_p\gamma^p. \tag{3.9a}$$

Here we use the abbreviated notation $C_\delta\cap B(m)$ for $C_\delta\cap(B(m)\times R^2)$.

Note that when $\varepsilon>0$, $(\times_{i=1}^n L_i)(\varepsilon,\mu_x)$ is continuous in all its arguments. Therefore

$$\begin{aligned}&E^{\bar\rho}\left(\sup_{\substack{x\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right|\right)\\&\quad=E^{\bar\rho}\left(\sup_{(x,\,\varepsilon,\,\varepsilon')\in C_\delta\cap B(m)}\left|\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right|\right).\end{aligned} \tag{3.10}$$

It follows from (3.9a), (3.10) and Theorem 2.9 that

$$E^{\bar\rho}\left(\sup_{\substack{x\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\mathop{\times}_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right|\right)\le C_p\gamma \tag{3.11}$$

for some constant C_p depending only on p.

Fix $y \in B(m/4)$. Using the fact that

$$\Big(\bigtimes_{j=1}^{n} L_j\Big)(\varepsilon, \mu_x, t)(\omega_1 + z_1, \dots, \omega_n + z_n) = \Big(\bigtimes_{j=1}^{n} L_j\Big)(\varepsilon, \mu_{x-z}, t)(\omega_1, \dots, \omega_n), \tag{3.12}$$

we see that for any $y' \in B(m/4)$

$$\begin{aligned} &E^{y}\left(\sup_{\substack{x\in B(m/4)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_x) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_x)\right|\right)\\ &\quad= E^{y'}\left(\sup_{\substack{x\in B(m/4)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_{x+y-y'}) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_{x+y-y'})\right|\right)\\ &\quad\le E^{y'}\left(\sup_{\substack{x\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_x) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_x)\right|\right). \end{aligned} \tag{3.13}$$

This shows that the first term in (3.13) does not depend on $y \in B(m/4)$. The measure $\bar\rho$ in (3.11) can be chosen with great generality. It only effects the constant C. Thus we choose a $\bar\rho$ which is supported on $B(m/4)$. With this choice of $\bar\rho$, keeping in mind the first sentences in this paragraph, we see that for all $y \in B(m/4)$,

$$E^{y}\left(\sup_{\substack{x\in B(m/4)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_x) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_x)\right|\right) \le C\gamma. \tag{3.14}$$

We now observe that for any $y \in B(m/16)$,

$$\begin{aligned} &E_A^{y_A}\left(\sup_{\substack{v,\,x\in B(m/16)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|E_{A^c}^{v_{A^c}}\left\{\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_x) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_x)\right\}\right|\right)\\ &\quad\le E_A^{y_A}\left(\sup_{v\in B(m/16)} E_{A^c}^{v_{A^c}}\left\{\sup_{\substack{x\in B(m/16)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_x) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_x)\right|\right\}\right)\\ &\quad= E_A^{y_A}\Bigg(\sup_{v\in B(m/16)} E_{A^c}^{y_{A^c}}\\ &\qquad\times\left\{\sup_{\substack{x\in B(m/16)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|\Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon, \mu_{x+v_{A^c}-y_{A^c}}) - \Big(\bigtimes_{i=1}^{n} L_i\Big)(\varepsilon', \mu_{x+v_{A^c}-y_{A^c}})\right|\right\}\Bigg) \end{aligned} \tag{3.15}$$

$$\leq E_A^{y_A}\left(E_{A^c}^{y_{A^c}}\left\{\sup_{\substack{x\in B(m/4)\\0<\varepsilon,\,\varepsilon'\leq\delta}}\left|\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right|\right\}\right)$$

$$= E^y\left(\sup_{\substack{x\in B(m/4)\\0<\varepsilon,\,\varepsilon'\leq\delta}}\left|\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right|\right).$$

Therefore, it follows from (3.14) that for any $y \in B(m/16)$,

$$E_A^{y_A}\left(\sup_{\substack{v,\,x\in B(m/16)\\0<\varepsilon,\,\varepsilon'\leq\delta}}\left|E_{A^c}^{v_{A^c}}\left\{\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon,\mu_x)-\left(\bigtimes_{i=1}^n L_i\right)(\varepsilon',\mu_x)\right\}\right|\right)\leq c\gamma. \tag{3.16}$$

Note that for any finite sets $D \subseteq B(m/16)$ and $D_\delta \subseteq (0, \delta]$,

$$\sup_{\substack{v,\,x\in D\\ \varepsilon,\,\varepsilon'\in D_\delta}}\left\{M_A(\varepsilon,\mu_x,t_A,v_{A^c})-M_A(\varepsilon',\mu_x,t_A,v_{A^c})\right\} \tag{3.17}$$

is a right continuous submartingale separately in each coordinate of t_A. Consequently, using (3.2) and the definition (3.3) of $M_A(\varepsilon, \mu_x, t_A, v_{A^c})$, it follows from (3.16) that

$$E_A^{y_A}\left(\sup_{t_A}\sup_{\substack{v,\,x\in D\\0<\varepsilon,\,\varepsilon'\in D_\delta}}\left\{M_A(\varepsilon,\mu_x,t_A,v_{A^c})-M_A(\varepsilon',\mu_x,t_A,v_{A^c})\right\}\right)\leq C\gamma, \tag{3.18}$$

where C is independent of the finite sets D and D_δ. By the monotone convergence theorem, (3.18) continues to hold with D and D_δ replaced by all the rational elements in $B(m/16)$, $(0, \delta]$. This enables us to complete the proof of Lemma 3.1 because it implies that for each $y \in (R^d)^n$ and $A \subseteq \{1, \dots, n\}$, $M_A(\varepsilon, \mu_x, t_A, v_{A^c})$ converges P^y almost surely rationally locally uniformly in $(x, v, t) \in (R^d)^n \times (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$.

As we discussed in the paragraph following the statement of Lemma 3.2, we have now established that $(\bigtimes_{j=1}^n L_j)(\varepsilon, \mu_x, t)$ converges almost surely locally uniformly in $(x, t) \in (R^d)^n \times (R_+)^n$ as $\varepsilon \to 0$. Hence,

$$\mathscr{L}(\mu_x, t) =_{\text{def}} \lim_{\varepsilon\to 0}\left(\bigtimes_{j=1}^n L_j\right)(\varepsilon,\mu_x,t) \tag{3.19}$$

is continuous almost surely. Because of the locally uniform convergence, it is clear that $\mathscr{L}(\mu, t)$ is a continuous additive functional. We now show that it has potential $(\bigtimes_{j=1}^n U_j^1)\mu(y_1, \dots, y_n)$; that is, that

$$E^{y_1,\dots,y_n}(\mathscr{L}(\mu,\infty,\dots,\infty)) = \int\int\prod_{j=1}^n u_j^1(y_j-x_j)\,d\mu(x_1,\dots,x_n) \tag{3.20}$$

for all $(y_1, \dots, y_n) \in (R^d)^n$. This shows that $\mathscr{L}(\mu_x, t)$ is equivalent to the process given in (1.11) and completes the proof of Theorem 1.2.

We now obtain (3.20). Note that by (3.14) and the dominated convergence theorem,

$$E^{y_1,\dots,y_n}\big(\mathscr{L}(\mu,\infty,\dots,\infty)\big) = \lim_{\varepsilon\to 0} E^{y_1,\dots,y_n}\Big\{\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\varepsilon,\mu,\infty,\dots,\infty)\Big\} \tag{3.21}$$

for all $y \in (R^d)^n$. Also it is easy to check that

$$\begin{aligned} &E^{y_1,\dots,y_n}\Big\{\Big(\mathop{\times}_{j=1}^{n} L_j\Big)(\varepsilon,\mu,\infty,\dots,\infty)\Big\} \\ &\quad= \int\Big\{\int \prod_{j=1}^{n} u_j^1(y_j - z_j) f_{\varepsilon, x_j}(z_j)\,dz_j\Big\}\,d\mu(x_1,\dots,x_n) \\ &\quad= \int\Big\{\int \prod_{j=1}^{n} u_j^1(y_j - z_j + x_j)\,d\mu(x_1,\dots,x_n)\Big\}\prod_{j=1}^{n} f_\varepsilon(z_j)\,dz_j. \end{aligned} \tag{3.22}$$

Since μ is a finite measure,

$$\int \prod_{j=1}^{n} u_j^1(y_j - z_j + x_j)\,d\mu(x_1,\dots,x_n) \in L^1(dz_1\dots dz_n). \tag{3.23}$$

Therefore,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int\Big\{\int \prod_{j=1}^{n} u_j^1(y_j - z_j + x_j)\,d\mu(x_1,\dots,x_n)\Big\}\prod_{j=1}^{n} f_\varepsilon(z_j)\,dz_j \\ &\quad= \int \prod_{j=1}^{n} u_j^1(y_j - x_j)\,d\mu(x_1,\dots,x_n) \end{aligned} \tag{3.24}$$

for almost all $(y_1,\dots,y_n) \in (R^d)^n$. Since both sides of (3.20) are excessive, we see that (3.24) holds for all $(y_1,\dots,y_n) \in (R^d)^n$. This completes the proof of Theorem 1.2. □

4. Isomorphism theorem for "near intersections" of a single Lévy process. In this section we obtain a different extension of Dynkin's isomorphism theorem which we use to prove Theorem 1.3. Even though $\{L^n(x,t;\mu), (x,t) \in (R^d)^n_{\neq} \times R^+\}$ is generated by a single Lévy process X, our method of proof requires that we deal with a mixture of functionals of several independent copies of X and their associated Gaussian chaoses.

For each subset $A \subseteq \{1,\dots,n\}$, $x = (x_1,\dots,x_n) \in (R^d)^n_{\neq}$, $\varepsilon > 0$ and $\mu \in \mathscr{G}_n^{2n}$, let

$$(L^A \times H^{A^c})(\varepsilon,\mu,x) =_{\text{def}} \int \prod_{i\in A} L_\lambda^{y+x_i,\varepsilon} \prod_{j\in A^c} H(y+x_j,\varepsilon)\,d\mu(y). \tag{4.1}$$

In the course of proving the next theorem we show that for each subset $A \subseteq \{1,\dots,n\}$, $(L^A \times H^{A^c})(\varepsilon,\mu,x)$ converges in L^2, as $\varepsilon \to 0$, for all $\mu \in \mathscr{G}_n^{2n}$

and $x \in (R^d)^n_{\neq}$. Let

$$(L^A \times H^{A^c})(\mu, x) =_{\text{def}} \lim_{\varepsilon\to 0}(L^A \times H^{A^c})(\varepsilon, \mu, x). \tag{4.2}$$

In particular,

$$\begin{aligned} L^n(\mu, x) &=_{\text{def}} L^{\{1,\dots,n\}}(\mu, x) \\ &= \lim_{\varepsilon\to 0}\int \prod_{i=1}^n L_\lambda^{y+x_i,\varepsilon}\, d\mu(y). \end{aligned} \tag{4.3}$$

In Theorem 1.3 all the processes are determined by a single Lévy process X with 1-potential u^1. Let $H(x,\varepsilon)$ be the second-order Gaussian chaos defined through u^1 as in (1.18)–(1.21) and (2.1)–(2.3). Let $H_1(x,\varepsilon),\dots,H_n(x,\varepsilon)$ denote independent copies of $H(x,\varepsilon)$.

Let λ_j denote independent mean-1 exponential random variables and let $X_1,\dots,X_n$, $j=1,\dots,n$ denote n independent copies of X. Let $L^{x,\varepsilon}_{j,t}$, $j = 1,\dots,n$ be as defined in (2.10).

We work with partitions $\bigcup_i A_i \bigcup_j B_j$ of $\{1,\dots,n\}$. To avoid complicated notation we write this as $\{1,\dots,n\} = \bigcup_{i=1}^n A_i \bigcup_{j=1}^n B_j$, even though some of the A_i and B_j are empty. For $x \in (R^d)^n_{\neq}$, and each partition $\{1,\dots,n\} = \bigcup_{i=1}^n A_i \bigcup_{j=1}^n B_j$ let

$$\begin{aligned} &\left(\bigtimes_{i=1}^n L_i^{A_i} \bigtimes_{j=1}^n H_j^{B_j}\right)(\varepsilon,\mu,x) \\ &\quad =_{\text{def}} \int \prod_{i=1}^n \prod_{k\in A_i} L_{i,\lambda_i}^{y+x_k,\varepsilon} \prod_{j=1}^n \prod_{l\in B_j} H_j(y+x_l,\varepsilon)\, d\mu(y) \end{aligned} \tag{4.4}$$

where $\varepsilon > 0$ and $\mu \in \mathscr{G}^{2n}$.

Similarly when $C_i \subset A_i$ we write

$$\begin{aligned} &\left(\bigtimes_{i=1}^n L_i^{C_i} \bigtimes_{i=1}^n H_i^{A_i - C_i} \bigtimes_{j=1}^n H_j^{B_j}\right)(\varepsilon,\mu,x) \\ &\quad =_{\text{def}} \int \prod_{i=1}^n \left(\prod_{\substack{k\in C_i \\ l\in A_i - C_i}} L_{i,\lambda_i}^{y+x_k,\varepsilon} H_i(y+x_l,\varepsilon)\right) \\ &\qquad \times \prod_{j=1}^n \prod_{m\in B_j} H_j(y+x_m,\varepsilon)\, d\mu(y), \end{aligned} \tag{4.5}$$

where $\varepsilon > 0$ and $\mu \in \mathscr{G}^{2n}$.

One can check that for each partition $\{1,\dots,n\} = \bigcup_{j=1}^n B_j$, the nth order Gaussian chaos $(\bigtimes_{j=1}^n H_j^{B_j})(\varepsilon,\mu,x)$, converges in L^2, as $\varepsilon \to 0$, for all $\mu \in \mathscr{G}^{2n}$ and $x \in (R^d)^n_{\neq}$.

In the course of proving the next theorem we show that for each partition $\{1, \dots, n\} = \bigcup_{i=1}^n A_i \bigcup_{j=1}^n B_j$, the process $(\times_{i=1}^n L_i^{A_i} \times_{j=1}^n H_j^{B_j})(\varepsilon, \mu, x)$ converges in L^2 as $\varepsilon \to 0$, for all $\mu \in \mathscr{G}^{2n}$ and $x \in (R^d)^n_{\neq}$. Let

$$\left(\mathop{\times}_{i=1}^n L_i^{A_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(\mu, x) =_{\text{def}} \lim_{\varepsilon\to 0}\left(\mathop{\times}_{i=1}^n L_i^{A_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(\varepsilon, \mu, x). \tag{4.6}$$

To unify the notation we set

$$\left(\mathop{\times}_{i=1}^n L_i^{A_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(0, \mu, x) = \left(\mathop{\times}_{i=1}^n L_i^{A_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(\mu, x). \tag{4.7}$$

The following isomorphism theorem is used in the proof of Theorem 1.3. The proof is similar to the proof of Theorem 2.8 and is omitted.

THEOREM 4.1. *Let $\mu \in \mathscr{G}^{2n}$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers and $\{x^i\}_{i=1}^\infty$ be a sequence of points in $(R^d)^n_{\neq}$. Let $\bigcup_{i=1}^n A_i \bigcup_{j=1}^n B_j$ be a partition of $\{1, \dots, n\}$ as described above and set $A = \bigcup_{i=1}^n A_i$, and similarly for B. Then, for any finite measures $\rho_j \in \mathscr{G}^1$, $j \in \{1, \dots, n\}$ and functions g_j with $g_j \cdot dx \in \mathscr{G}^1$, $j \in \{1, \dots, n\}$ and $\mathscr{C}$ measurable nonnegative function F on R^∞,*

$$\begin{aligned} &E_{\bar G} E_{\bar\lambda}^{\bar\rho}\Bigg(F\Bigg(\sum_{C_i \subseteq A_i} \frac{1}{2^{n-|C|}}\left(\mathop{\times}_{i=1}^n L_i^{C_i} \mathop{\times}_{i=1}^n H_i^{A_i - C_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(\varepsilon_\cdot, \mu, x^\cdot)\Bigg) \\ &\qquad\qquad \times \prod_{j\in A} g_j(X_j(\lambda_j))\Bigg) \\ &\quad = E_{\bar G}\Bigg(F\Bigg(\frac{1}{2^n}\left(\mathop{\times}_{i=1}^n H_i^{A_i} \mathop{\times}_{j=1}^n H_j^{B_j}\right)(\varepsilon_\cdot, \mu, x^\cdot)\Bigg) \prod_{j\in A} G_{j,\rho_j} G_{j, g_j\cdot dx}\Bigg). \end{aligned} \tag{4.8}$$

where $C = \bigcup_{i=1}^n C_i$ and $\mathscr{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

Let $H^n(\varepsilon, \mu, x)$ be given by (4.1) with $A^c = \{1, \dots, n\}$. Note that when we consider only a single Lévy process X in Theorem 4.1 and B is empty we can write (4.8) as

$$\begin{aligned} &E_G E_\lambda^\rho\Bigg(F\Bigg(\sum_{C \subseteq \{1,\dots,n\}} \frac{1}{2^{|C^c|}}(L^C \times H^{C^c})(\varepsilon_\cdot, \mu, x^\cdot)\Bigg) g(X(\lambda))\Bigg) \\ &\quad = E_G\Bigg(F\Bigg(\frac{1}{2^n} H^n(\varepsilon_\cdot, \mu, x^\cdot)\Bigg) G_\rho G_{g\cdot dx}\Bigg). \end{aligned} \tag{4.9}$$

Note that by definition,

$$L^n(x, \lambda; \mu) = L^n(\mu, x). \tag{4.10}$$

Consistent with the above notation we let $\{A_i\}_{i=1}^n$ be a partition of $\{1, \dots, n\}$. As above it is possible that some of the A_i are empty. Recall $A = \bigcup_{i=1}^n A_i$.

Just as we obtained Theorem 2.8 from Theorem 2.9 we would like to extract from Theorem 4.1 an inequality that separates the intersection functional from the Gaussian chaoses. The situation is more complex here. We begin with the following lemma. Let $\|\cdot\|$ be a norm on ℓ^∞. We continue to use the notation described just before Theorem 2.9.

LEMMA 4.1. *Let $\mu \in \mathscr{G}^{2n}$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers and $\{x^i\}_{i=1}^\infty$ be a sequence of points in $(R^d)^n_{\neq}$. There exists a constant p' depending only on u_j^1, $j=1,\dots,n$ and a constant K depending only on n, ρ_j and u_j^1, $j=1,\dots,n$ and p', such that*

$$
\begin{aligned}
E_{\bar\lambda}^{\bar\rho}\left\|\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon,\mu,x)\right\| &\le K\left(E_{\bar G}\left\|\left(\mathop{\times}_{i=1}^n H_i^{A_i}\right)(\varepsilon,\mu,x)\right\|^{p'}\right)^{1/p'} \\
&\quad + \sum_{\substack{C_i\subseteq A_i\\|C|<n}} E_{\bar\lambda}^{\bar\rho}\left\|\left(\mathop{\times}_{i=1}^n L_i^{C_i}\right)(\varepsilon,\mu,x)\mathscr{T}_{A_i-C_i}(x,\varepsilon)\right\|.
\end{aligned}
\tag{4.11}
$$

Here $(\times_{i=1}^n L_i^{C_i})(\varepsilon,\mu,x)$ is defined as in (4.1) but with $x\in(R^d)^{|C|}_{\neq}$ and

$$
\mathscr{T}_{A_i-C_i}(x,\varepsilon)=E_{\bar G}\prod_{i=1}^n\prod_{l\in A_i-C_i}H_i(y+x_l,\varepsilon),
$$

and does not depend on y.

PROOF. Using Theorem 4.1 with B empty and proceeding as in the proof of Theorem 2.9 we have

$$
\begin{aligned}
&E_{\bar G}E_{\bar\lambda}^{\bar\rho}\left\|\sum_{C_i\subseteq A_i}\frac{1}{2^{n-|C|}}\left(\mathop{\times}_{i=1}^n L_i^{C_i}\mathop{\times}_{i=1}^n H_i^{A_i-C_i}\right)(\varepsilon,\mu,x)\right\| \\
&\qquad\le K\left(E_{\bar G}\left\|\left(\mathop{\times}_{i=1}^n H_i^{A_i}\right)(\varepsilon,\mu,x)\right\|^{p'}\right)^{1/p'}.
\end{aligned}
\tag{4.12}
$$

We write the first line of (4.12) in the form

$$
E_{\bar G}E_{\bar\lambda}^{\bar\rho}\left\|\sum_{C_i\subseteq A_i}Z_{C_i}\right\| = E_{\bar G}E_{\bar\lambda}^{\bar\rho}\left\|\tilde Z+\sum_{\substack{C_i\subseteq A_i\\|C|<n}}Z_{C_i}\right\|,
\tag{4.13}
$$

where

$$
\tilde Z=\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon,\mu,x),
\tag{4.14}
$$

that is, this is the term in the sum in which $|C|=n$ and so $C_i=A_i$ for all $i=1,\dots,n$. It follows from (4.13) that

$$
E_{\bar\lambda}^{\bar\rho}\|\tilde Z\|\ge E_{\bar G}E_{\bar\lambda}^{\bar\rho}\left\|\sum_{C_i\subseteq A_i}Z_{C_i}\right\|-\sum_{\substack{C_i\subseteq A_i\\|C|<n}}E_{\bar\lambda}^{\bar\rho}\|E_{\bar G}Z_{C_i}\|.
\tag{4.15}
$$

Combining (4.12)–(4.15) we get

$$E_{\bar{\lambda}}^{\bar{\rho}}\left\|\left(\bigtimes_{i=1}^{n} L_i^{A_i}\right)(\varepsilon,\mu,x)\right\| \le K\left(E_{\bar{G}}\left\|\left(\bigtimes_{i=1}^{n} H_i^{A_i}\right)(\varepsilon,\mu,x)\right\|^{p'}\right)^{1/p'} + \sum_{\substack{C_i\subseteq A_i\\ |C|<n}} E_{\bar{\lambda}}^{\bar{\rho}}\|E_{\bar{G}} Z_{C_i}\|, \tag{4.16}$$

which is (4.11). This completes the proof of Lemma 4.1. □

One can continue to develop the last term in (4.11) recursively depending on the nature of the norm. However, since $\mathscr{T}_{A_i-C_i}(x,\varepsilon)$ is independent of y we can write

$$\mathscr{T}_{A_i-C_i}(x,\varepsilon) = \int \mathscr{T}_{A_i-C_i}(x,\varepsilon)\,d\mu(y). \tag{4.17}$$

Consequently, for any norm, we have

$$\left\|\mathscr{T}_{A_i-C_i}(x,\varepsilon)\right\| \le E_{\bar{G}}\left\|\bigtimes_{i=1}^{n} H^{A_i-C_i}(\varepsilon,\mu,x)\right\|. \tag{4.18}$$

We now specify some particular norms. For a function $h(x,\varepsilon)$ we set

$$\|h(x,\varepsilon)\| =_{\text{def}} \sup_{x,\varepsilon,\varepsilon'} |h(x,\varepsilon) - h(x,\varepsilon')| \tag{4.19}$$

and

$$\|h(x,\varepsilon)\| =_{\text{def}} \sup_{x,\varepsilon} |h(x,\varepsilon)|. \tag{4.20}$$

For $\tau > 0$ let

$$K_\tau^r =_{\text{def}} \left\{x = (x_1,\dots,x_r) \in (R^d)^r : |x_i - x_j| \ge \tau,\ \forall\, i,j; i \ne j\right\}. \tag{4.21}$$

For $\delta > 0$, let $C_{\delta,\tau}$ be a countable dense subset of $K_\tau^r \times (0,\delta]^2$. In what follows we let $B(m)$ denote the ball of radius m in $(R^d)^r$. We do not denote the dimension r in these last two definitions; it will be clear in the context in which they appear.

For $1 \le r \le n$ let $\{A_{i,r}\}_{i=1}^r$ be a partition of $\{1,\dots,r\}$. As above it is possible that some of the $A_{i,r}$ are empty. Also, as above, when $r = n$, and it is not confusing, we use A_i instead of $A_{i,n}$.

The next lemma is used in the proof of Theorem 1.3.

LEMMA 4.2. *Assume that*

$$\left\{\left(\bigtimes_{i=1}^{r} H_i^{A_{i,r}}\right)(\mu,x); x \in (R^d)^r_{\ne}\right\} \tag{4.22}$$

is continuous almost surely for each partition $\{A_{i,r}\}_{i=1}^r$ *of* $\{1,\dots,r\}$ *for all* $1 \le r \le n$. *Then for all* $\gamma > 0$ *we can find a* $\delta > 0$ *such that*

$$E_{\bar{\lambda}}^{\bar{\rho}}\left(\sup_{(x,\varepsilon,\varepsilon')\in C_{\delta,\tau}\cap B(m)}\left|\left(\bigtimes_{i=1}^{r} L_i^{A_{i,r}}\right)(\varepsilon,\mu,x) - \left(\bigtimes_{i=1}^{r} L_i^{A_{i,r}}\right)(\varepsilon',\mu,x)\right|\right) \le \gamma \tag{4.23}$$

for all partitions $\{A_{i,r}\}_{i=1}^r$ *of* $\{1,\dots,r\}$ *for all* $1 \le r \le n$.

PROOF. Let $\|\cdot\|_\delta$ and $|||\cdot|||_\delta$ denote the norms defined in (4.19) and (4.20) but with $\sup_{x,\varepsilon,\varepsilon'}$ replaced by $\sup_{(x,\varepsilon,\varepsilon')\in C_{\delta,\tau}\cap B(m)}$ and $\sup_{x,\varepsilon}$ replaced by $\sup_{(x,\varepsilon)\in C_{\delta,\tau}\cap B(m)}$. By hypothesis,

$$\left(\mathop{\times}_{i=1}^{r} H_i^{A_{i,r}}\right)(\mu, x)_\varepsilon =_{\text{def}} \int \left(\mathop{\times}_{i=1}^{r} H_i^{A_{i,r}}\right)(\mu, y_1, \dots, y_r) \prod_{i=1}^{r} f_\varepsilon(x_i - y_i)\, dy_i \tag{4.24}$$

is continuous in $x \in K_\tau^r$ almost surely for $\varepsilon > 0$ sufficiently small and converges to $(\times_{i=1}^n H_i^{A_{i,r}})(\mu, x)$ locally uniformly in $x \in K_\tau^r$ almost surely as $\varepsilon \to 0$. Consequently for any $\gamma' > 0$, we can find $\delta > 0$ and a countable dense subset $C_{\delta,\tau}$ of $K_\tau^r \times (0, \delta]^2$ such that

$$E_{\bar G}\left\|\left(\mathop{\times}_{i=1}^{r} H_i^{A_{i,r}}\right)(\varepsilon, \mu, x)\right\|_\delta \le \gamma' \tag{4.25}$$

and

$$E_{\bar G}\left|\left|\left|\left(\mathop{\times}_{i=1}^{r} H_i^{A_{i,r}}\right)(\varepsilon, \mu, x)\right|\right|\right|_\delta < \infty \tag{4.26}$$

for all partitions $\{A_{i,r}\}_{i=1}^r$ of $\{1, \dots, r\}$ for all $1 \le r \le n$.

Using Lemma 4.1, (4.18)–(4.20), (4.25) and (4.26) we see that the left-hand side of (4.11) is less than or equal to

$$\begin{aligned} &C\gamma' + \sum_{\substack{C_i \subseteq A_i \\ |C|<n}} \Bigg(\left\|\mathscr{T}_{A_i - C_i}(x, \varepsilon)\right\|_\delta \left|\left|\left|\left(\mathop{\times}_{i=1}^{r} L_i^{C_i}\right)(\varepsilon, \mu, x)\right|\right|\right|_\delta \\ &\qquad + \left|\left|\left|\mathscr{T}_{A_i - C_i}(x, \varepsilon)\right|\right|\right|_\delta \left\|\left(\mathop{\times}_{i=1}^{r} L_i^{C_i}\right)(\varepsilon, \mu, x)\right\|_\delta \Bigg) \\ &\le C\gamma' + C\gamma' \sum_{\substack{C_i \subseteq A_i \\ |C|<n}} \left|\left|\left|\left(\mathop{\times}_{i=1}^{n} L_i^{C_i}\right)(\varepsilon, \mu, x)\right|\right|\right|_\delta \\ &\qquad + C \sum_{\substack{C_i \subseteq A_i \\ |C|<n}} \left\|\left(\mathop{\times}_{i=1}^{n} L_i^{C_i}\right)(\varepsilon, \mu, x)\right\|_\delta . \end{aligned} \tag{4.27}$$

Using this and (4.11) we can use a proof by induction to obtain (4.23) . Note that we use the fact that $(\times_{i=1}^n L_i^{C_i})(1, \mu, x) < \infty$ and

$$\left\|\left(\mathop{\times}_{i=1}^{n} L_i^{C_i}\right)(\varepsilon, \mu, x)\right\|_\delta < \infty \text{ implies that } \left|\left|\left|\left(\mathop{\times}_{i=1}^{n} L_i^{C_i}\right)(\varepsilon, \mu, x)\right|\right|\right|_\delta < \infty.$$

REMARK 4.1. The continuity condition (4.22) is the critical condition used in the proof of Theorem 1.3. We now explain how it is implied by the continuity of the chaoses $\mathscr{H}_{3,r}$. It follows from Theorem 1.1 and (2.13) in [3] that

$$\left\{\left(\mathop{\times}_{i=1}^{r} \widetilde{H}_i^{A_{i,r}}\right)(\mu, x);\ x \in (R^d)^r_{\neq}\right\} \tag{4.28}$$

is continuous almost surely for each partition $\{A_{i,r}\}_{i=1}^r$ of $\{1,\dots,r\}$ for all $1 \le r \le n$, where

$$\left(\mathop{\times}_{i=1}^{r} \widetilde{H}_i^{A_{i,r}}\right)(\mu, x) =_{\text{def}} \lim_{\varepsilon\to 0} \int \prod_{i=1}^{r} \prod_{l\in A_{i,r}} H_{i,\, y+x_l,\, \varepsilon}\, d\mu(y) \tag{4.29}$$

[See (1.22)]. To be more explicit, Theorem 1.1 in [3] implies (4.29), in the case $r = 1$. The formula (2.13) in [3] is the key step in the proof of Theorem 1.1 in [3]. To obtain (4.29) as stated, one follows the proof of Theorem 1.1 in [3] but uses products of independent processes and products of the corresponding terms in (2.13) in [3].

The process $(\times_{i=1}^r H_i^{A_{i,r}})(\mu, x)$ is defined as in (4.29), but with $H_{i,\,y+x_l,\,\varepsilon}$ replaced by $H_i(y + x_l, \varepsilon)$. This is analogous the situation considered in Lemma 2.2. As in that lemma we can show that $(\times_{i=1}^r \widetilde{H}_i^{A_{i,r}})(\mu, x)$ and $(\times_{i=1}^r H_i^{A_{i,r}})$ (μ, x) are equivalent stochastic processes. Therefore, the hypotheses of Theorem 1.3 implies the continuity condition in (4.22).

5. Continuity theorem for "near intersections" of a single Lévy process.

PROOF OF THEOREM 1.3. The proof follows along the lines of the proof of Theorem 1.2. As in Theorem 1.2 we prove this theorem for Y which is an exponentially killed version of the Lévy process X.

Let $\bigcup_{i=1}^m A_i$ be a partition of $\{1,\dots,n\}$. For $x = (x_1,\dots,x_n) \in (R^d)^n_{\neq}$ and $\varepsilon > 0$ we set

$$\left(\mathop{\times}_{i=1}^{m} L_i^{A_i}\right)(\varepsilon, \mu, x; t_1,\dots,t_m) =_{\text{def}} \int \prod_{i=1}^{m} \prod_{k\in A_i} L_{i,\,t_i}^{y+x_k,\varepsilon}\, d\mu(y) \tag{5.1}$$

[see (2.10)]. For each $v \in (R^d)^n$ and $y \in R^d$ consider the $\mathscr{F}_{1,t}$ martingale

$$\begin{aligned} &M_{A_1,\dots,A_m}(\varepsilon, \mu, x, v; t) \\ &\quad =_{\text{def}} E_1^y\left(E_{2,\dots,m}^{v_2,\dots,v_m}\left\{\left(\mathop{\times}_{i=1}^{m} L_i^{A_i}\right)(\varepsilon, \mu, x; \infty,\dots,\infty)\right\}\Big|\, \mathscr{F}_{1,t}\right). \end{aligned} \tag{5.2}$$

(Recall that the probability space here is generated by m independent copies $\{Y_1,\dots,Y_m\}$ of Y. $\mathscr{F}_{1,t}$ is the σ-field generated by Y_1.) We see below that, as the notation indicates, $M_{A_1,\dots,A_m}(\varepsilon, \mu, x, v; t)$ does not depend on y.

LEMMA 5.1. *Under the conditions of Theorem* 1.3, *for each partition* $\{1, \dots, n\} = \bigcup_{i=1}^{m} A_i$, *the martingale* $M_{A_1,\dots,A_m}(\varepsilon, \mu, x, v; t)$ *converges almost surely rationally locally uniformly in* $(x, v; t) \in (R^d)^n_{\neq} \times (R^d)^n \times R_+$ *as* $\varepsilon \to 0$.

Before giving the proof of Lemma 5.1 we show that it implies that $L^n(\varepsilon, \mu, x; t)$ converges almost surely locally uniformly in $(x; t) \in (R^d)^n_{\neq} \times R_+$ as $\varepsilon \to 0$. This is a consequence of the following lemma which actually shows more.

LEMMA 5.2. *Under the conditions of Theorem* 1.3, *for each partition* $\{1, \dots, n\} = \bigcup_{i=1}^{m} A_i$, $E^{v_2,\dots,v_m}_{2,\dots,m}\{(\times_{i=1}^{m} L_i^{A_i})(\varepsilon, \mu, x; t, \infty, \dots, \infty)\}$ *converges almost surely locally uniformly in* $(x, v, t) \in (R^d)^n_{\neq} \times (R^d)^n \times R_+$ *as* $\varepsilon \to 0$.

Note that when $m = 1$, so that $A_1 = \{1, \dots, n\}$ and all other A_i are empty, we have

$$E^{v_2,\dots,v_m}_{2,\dots,m}\left\{\left(\mathop{\times}_{i=1}^{m} L_i^{A_i}\right)(\varepsilon, \mu, x; t, \infty, \dots, \infty)\right\} = L^{n,\varepsilon}(x, t{:}\, \mu) \tag{5.3}$$

[see (1.12)]. Thus Lemma 5.2 does show that $L^n(\varepsilon, \mu, x; t)$ converges almost surely locally uniformly in $(x; t) \in (R^d)^n_{\neq} \times R_+$ as $\varepsilon \to 0$.

PROOF OF LEMMA 5.2. Since $E^{v_2,\dots,v_m}_{2,\dots,m}\{(\times_{i=1}^{m} L_i^{A_i})(\varepsilon, \mu, x; t, \infty, \dots, \infty)\}$ is clearly continuous in $(\varepsilon, x, v, t) \in (0, 1] \times (R^d)^n_{\neq} \times (R^d)^n \times R_+$, it suffices to show that $E^{v_2,\dots,v_m}_{2,\dots,m}\{(\times_{i=1}^{m} L_i^{A_i})(\varepsilon, \mu, x; t, \infty, \dots, \infty)\}$ converges almost surely rationally locally uniformly in $(x, v, t) \in (R^d)^n_{\neq} \times (R^d)^n \times R_+$ as $\varepsilon \to 0$. We proceed by induction on $|A_1|$. When $A_1 = \varnothing$,

$$\begin{aligned}
&E^{v_2,\dots,v_m}_{2,\dots,m}\left\{\left(\mathop{\times}_{i=1}^{m} L_i^{A_i}\right)(\varepsilon, \mu, x; t, \infty, \dots, \infty)\right\} \\
&\quad = E^{v_2,\dots,v_m}_{2,\dots,m}\left\{\left(\mathop{\times}_{i=2}^{m} L_i^{A_i}\right)(\varepsilon, \mu, x; \infty, \dots, \infty)\right\} \\
&\quad = M_{\varnothing, A_2,\dots,A_m}(\varepsilon, \mu, x, v; t) \\
&\quad = M_{A_2,\dots,A_m}(\varepsilon, \mu, x, v; 0).
\end{aligned} \tag{5.4}$$

Therefore, when $A_1 = \varnothing$, the assertion of this lemma is given by Lemma 5.1.

We now describe the induction step. Assume that the statement in this lemma is true for all $A_1 \subseteq \{1, \dots, n\}$ with $|A_1| < k$, and choose some $A_1 \subseteq \{1, \dots, n\}$ with $|A_1| = k$. Using additivity and the Markov property, we note that for any $y \in R^d$,

$$
\begin{aligned}
&M_{A_1,\dots,A_m}(\varepsilon,\mu,x,v;t)\\
&\quad =_{\text{def}} E_1^y\left(E_{2,\dots,m}^{v_2,\dots,v_m}\left\{\left(\mathop{\times}_{i=1}^m L_i^{A_i}\right)(\varepsilon,\mu,x;\infty,\dots,\infty)\right\}\bigg|\mathscr{F}_{1,t}\right)\\
&\quad = E_{1,2,\dots,m}^{y,v_2,\dots,v_m}\left\{\left(\mathop{\times}_{i=1}^m L_i^{A_i}\right)(\varepsilon,\mu,x;\infty,\dots,\infty)\bigg|\mathscr{F}_{1,t}\right\}\\
&\quad = \sum_{B\subseteq A_1}\int\prod_{i\in B}L_{1,t}^{z+x_i,\varepsilon}E_1^y\left(\prod_{j\in A_1-B}L_{1,\infty}^{z+x_j,\varepsilon}\circ\theta_{1,t}\Big|\mathscr{F}_{1,t}\right)\\
&\qquad\qquad\times E_{2,\dots,m}^{v_2,\dots,v_m}\left(\prod_{i=2}^m\prod_{k\in A_i}L_{i,t_i}^{y+x_k,\varepsilon}\right)d\mu(z)\\
&\quad = \sum_{B\subseteq A_1}\int\prod_{i\in B}L_{1,t}^{z+x_i,\varepsilon}E_1^{Y_1(t)}\left(\prod_{j\in A_1-B}L_{1,\infty}^{z+x_j,\varepsilon}\right)\\
&\qquad\qquad\times E_{2,\dots,m}^{v_2,\dots,v_m}\left(\prod_{i=2}^m\prod_{k\in A_i}L_{i,t_i}^{y+x_k,\varepsilon}\right)d\mu(z)\\
&\quad = \sum_{B\subseteq A_1}E_{2,\dots,m+1}^{v_2,\dots,v_m,Y_1(t)}\left\{\left(L_1^B\mathop{\times}_{i=2}^m L_i^{A_i}\times L_{m+1}^{A_1-B}\right)(\varepsilon,\mu,x;t,\infty,\dots,\infty)\right\}.
\end{aligned}\tag{5.5}
$$

Note that when $B = A_1$,

$$
\begin{aligned}
&E_{2,\dots,m+1}^{v_2,\dots,v_m,Y_1(t)}\left\{\left(L_1^B\mathop{\times}_{i=2}^m L_i^{A_i}\times L_{m+1}^{A_1-B}\right)(\varepsilon,\mu,x;t,\infty,\dots,\infty)\right\}\\
&\qquad = E_{2,\dots,m}^{v_2,\dots,v_m}\left\{\left(\mathop{\times}_{i=1}^m L_i^{A_i}\right)(\varepsilon,\mu,x;t,\infty,\dots,\infty)\right\}.
\end{aligned}\tag{5.6}
$$

Since all other terms in the last equality of (5.5) are of the form

$$
E_{2,\dots,m+1}^{v_2,\dots,v_m,v_{m+1}}\left\{\left(\mathop{\times}_{i=1}^{m+1} L_i^{B_i}\right)(\varepsilon,\mu,x;t,\infty,\dots,\infty)\right\}
$$

with $|B_1| < |A_1| = k$, the induction step is completed using Lemma 5.1. This completes the proof of Lemma 5.2. □

To complete the proof that $L^n(\varepsilon,\mu,x;t)$ converges almost surely locally uniformly in $(x;t)\in(R^d)^n_{\neq}\times R_+$ as $\varepsilon\to 0$ it only remains to prove Lemma 5.1. Here we use the isomorphism theorem in the form of Lemma 4.2.

PROOF OF LEMMA 5.1. By Remark 4.1, the hypotheses of Theorem 1.3 enable us to use Lemma 4.2 to show that for each partition $\{A_i\}_{i=1}^n$ of $\{1,\dots,n\}$, and for any $\gamma > 0$, we can find a $\delta > 0$, such that

$$
E^{\bar\rho}\left(\sup_{(x,\varepsilon,\varepsilon')\in C_{\delta,\tau}\cap B(m)}\left|\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon,\mu,x)-\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon',\mu,x)\right|\right)\le\gamma \tag{5.7}
$$

Following the arguments in (3.10) and (3.12)–(3.15) and readjusting m, we see that for any $y \in B(m)$ and $\gamma > 0$ we can find $\delta > 0$ such that

$$E^y\left(\sup_{\substack{x\in K^n_\tau;\, x,\, v\in B(m)\\ 0<\varepsilon,\,\varepsilon'\le\delta}}\left|E^{v_2,\ldots,v_n}_{2,\ldots,n}\left\{\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon,\mu,x)-\left(\mathop{\times}_{i=1}^n L_i^{A_i}\right)(\varepsilon',\mu,x)\right\}\right|\right) \le C\gamma. \tag{5.8}$$

Since for any finite sets $D \subseteq K^A_\tau \cap B(m)$, $D' \subseteq B(m)$ and $D_\delta \subseteq (0,\delta]$,

$$\sup_{\substack{x\in D,\, v\in D'\\ \varepsilon,\,\varepsilon'\in D_\delta}}\left\{M_{A_1,\ldots,A_n}(\varepsilon,\mu,x,v,t)-M_{A_1,\ldots,A_n}(\varepsilon',\mu,x,v,t)\right\} \tag{5.9}$$

is a right continuous submartingale in t, we have from (5.8) that

$$E^y\left(\sup_{t\ge 0}\sup_{\substack{x\in D,\, v\in D'\\ 0<\varepsilon,\,\varepsilon'\in D_\delta}}\left\{M_{A_1,\ldots,A_n}(\varepsilon,\mu,x,v,t)-M_{A_1,\ldots,A_n}(\varepsilon',\mu,x,v,t)\right\}\right) \le C\gamma, \tag{5.10}$$

where C is independent of the finite sets D, D' and D_δ. Therefore, by monotone convergence, (5.10) continues to hold with D, D' and D_δ replaced by all the rational elements in $K^n_\tau \cap B(m)$, $B(m)$ and $(0,\delta]$, respectively. This proves that $M_{A_1,\ldots,A_n}(\varepsilon,\mu,x,v,t)$ converge almost surely rationally locally uniformly in $(x,v,t) \in K^A_\tau \times (R^d)^n \times R_+$ as $\varepsilon \to 0$ for each partition $\{1,\ldots,n\} = \bigcup_{i=1}^n A_i$. Since this holds for all $\tau > 0$, Lemma 5.1 is proved. □

We have now shown that $L^{n,\varepsilon}(x,t;\mu)$ converges almost surely locally uniformly in $(x,t) \in (R^d)^n_{\neq} \times R_+$ as $\varepsilon \to 0$. Recalling (1.13), we see that Theorem 1.3 is proved. □

6. Continuity of Gaussian chaoses. In this section we explain how Theorems 1.4–1.6 follow from the results in [3]. Considering Theorems 1.1–1.3, it suffices to obtain sufficient conditions for the continuity almost surely of the three classes of Gaussian chaos processes $\mathscr{H}_1$, $\mathscr{H}_2$ and $\{\mathscr{H}_{3,r}; 1 \le r \le 2n\}$. We do this by associating with $\mathscr{H}_1$ and $\mathscr{H}_2$ corresponding decoupled Gaussian chaoses which have the property that the continuity of the decoupled chaoses implies the continuity of $\mathscr{H}_1$ and $\mathscr{H}_2$. The processes $\{\mathscr{H}_{3,r}; 1 \le r \le 2n\}$ are already decoupled. We then use results in [3] to show that the decoupled chaoses are continuous almost surely.

Corresponding to the Gaussian chaoses defined in 1′ and 2′ we define two types of decoupled Gaussian chaos processes. Recall the Gaussian process defined in (1.19). Let $\widetilde{G}_{j,x,\delta}$ be an independent copy of $G_{j,x,\delta}$.

1″. Let $x \in R^d$ and let $\mu \in \mathscr{G}^{2,n}$,

$$\mathscr{D}_1(x;\mu) =_{\text{def}} \lim_{\delta\to 0} \int \prod_{j=1}^{n} G_{j,x+y,\delta}\widetilde{G}_{j,x+y,\delta}\,d\mu(y). \tag{6.1}$$

This limit exists in L^2 of the probability space. Note that $\mathscr{D}_1(x;\mu) = \mathscr{D}_1(0;\mu_x)$.

2″. Let $x \in (R^d)^n$ and write it as $(x_1,\dots,x_n)$. Let $\mu \in \mathscr{G}_n^{2,n}$,

$$\mathscr{D}_2(x;\mu) =_{\text{def}} \lim_{\delta\to 0} \int \prod_{j=1}^{n} G_{j,x_j+y_j,\delta}\widetilde{G}_{j,x_j+y_j,\delta}\,d\mu(y). \tag{6.2}$$

This limit exists in L^2 of the probability space. Note that $\mathscr{D}_2(x;\mu) = \mathscr{D}_2(0;\mu_x)$.

The processes $\{\mathscr{H}_{3,r}; 1 \le r \le 2n\}$ are already decoupled. In keeping with the spirit of 1′ and 2′ we could have given (4.22) as the hypothesis of Theorem 1.3 but we did not want to introduce such complex notation so early in the paper. The reason we use $\mathscr{H}_1$ and $\mathscr{H}_2$ in the hypotheses of Theorems 1.1 and 1.2, rather than the corresponding decoupled chaoses, is because we only know that the continuity of the corresponding decoupled chaoses is a sufficient condition for the continuity of $\mathscr{H}_1$ and $\mathscr{H}_2$. Thus Theorems 1.1 and 1.2 may be stronger as stated and perhaps they give necessary and sufficient conditions and similarly for Theorem 1.3 with (4.22) as hypothesis. It seems very difficult to resolve these speculations.

PROOF OF THEOREM 1.4. By Theorem 1.3 it suffices to show that

$$\{\mathscr{H}_{3,r}(x;\mu),\ x \in (R^d)^n_{\neq}\}$$

is continuous almost surely. Theorem 1.3 in [3] shows that (i) of Theorem 1.4 in this paper is a sufficient condition for this to be the case.

In order to prove Theorem 1.4 under condition (ii) consider

$$\mathscr{H}_{3,r}(x_i;\mu) = \lim_{\delta\to 0} \int \prod_{j=1}^{r} G^{(j)}_{x_i+y,\delta}\,d\mu(y). \tag{6.3}$$

By (3.1) in [3],

$$\begin{aligned}\Big(E\big(\mathscr{H}_{3,r}(x;\mu) - \mathscr{H}_{3,r}(y;\mu)\big)^2\Big)^{1/2} \\ \le \sum_{i=1}^{r}\Big(E\big(\mathscr{H}_{3,r}(x_i;\mu) - \mathscr{H}_{3,r}(y_i;\mu)\big)^2\Big)^{1/2}.\end{aligned} \tag{6.4}$$

This shows that the metric entropy, with respect to the L^2 metric of $\mathscr{H}_{3,r}(x;\mu)$, of $([0,1]^d)^r$, is bounded by the rth power of the metric entropy, with respect to the L^2 metric of $\mathscr{H}_{3,r}(x_1;\mu)$, of $[0,1]^d$. Since we take the logarithm of the metric entropy in (1.29), we see that if (1.29) is satisfied for $\mathscr{H}_{3,r}(x_1;\mu)$ then it is satisfied for $\mathscr{H}_{3,r}(x;\mu)$. Corollary 4.2 in [3] and its proof show that (1.29) is satisfied for $\mathscr{H}_{3,r}(x_1;\mu)$ when (4.12) in [3] holds. This last condition is implied by (1.36) in this paper.

PROOF OF THEOREM 1.5. By Theorem 1.1 it suffices to show that $\{\mathscr{H}_1(x;\mu), x \in R^d\}$ is continuous almost surely. Processes such as these are studied in [3], but only in the case when all the chaoses $H_{j,x+y,\delta}$ are identically distributed, that is, when the 1-potentials u_j^1 are all equal. See (1.31) and (1.32) in [3] and note that in the notation of [3],

$$\mathscr{H}_1(x;\mu) = \mathfrak{G}^{2,\mathrm{dec}}_{\overline{2n},1,0,\mu}(\bar{x}), \tag{6.5}$$

where $\overline{2n} = \overbrace{(2,\dots,2)}^{n\text{-terms}}$ and all $x_{j,p}$, $p = 1, 2$, $j = 1, \dots, n$ are equal in (1.31) of reference [3], so that $\bar{x} = (x, \dots, x)$. In this case, that is, when the $H_{j,x+y,\delta}$ are identically distributed, it follows from the proof of Theorem 1.4 in [3] that $\{\mathscr{H}_1(x;\mu), x \in (R^d)\}$ is continuous almost surely if $\{\mathscr{D}_1(x;\mu), x \in R^d\}$ is continuous almost surely. [Note that $\{\mathscr{D}_1(x;\mu), x \in R^d\}$ is actually the restriction of $\mathfrak{G}^{\mathrm{dec}}_{\overline{2n},1,0,\mu}(x_1,\dots,x_{2n})$, in [3] to the diagonal of $(R^d)^{2n}$. The proof of Theorem 1.4 in [3] can be carried out in this case also.] More significantly, the proof of Theorem 1.4 in [3] also goes through when the $H_{j,x+y,\delta}$ are not identically distributed.

To prove Theorem 1.5 it remains to show that $\{\mathscr{D}_1(x;\mu), x \in R^d\}$ is continuous almost surely. Theorem 1.5(i) follows from the proof of Theorem 1.3 in [3] where one replaces $(u^1(\cdot))^{2n}$ by $\prod_{j=1}^n (u_j^1(\cdot))^2$. Theorem 1.5(ii) follows similarly from Corollary 4.2 in [3].

PROOF OF THEOREM 1.6. By Theorem 1.2 it suffices to show that $\{\mathscr{H}_2(x;\mu), x \in (R^d)^n\}$ is continuous almost surely. Processes such as these are studied in [3]. See (1.36) in [3] and note that in the notation of [3],

$$\mathscr{H}_2(x;\mu) = \mathfrak{G}^{n,\mathrm{dec}}_{2\times n,0,1,\mu}(x). \tag{6.6}$$

It follows from Theorem 1.5 in [3] that $\{\mathscr{H}_2(x;\mu), x \in (R^d)^n\}$ is continuous almost surely if $\{\mathscr{D}_2(x;\mu), x \in (R^d)^n\}$ is continuous almost surely. That this is implied by Theorem 1.6 in this paper follows from Theorem 1.6 in [3]. □

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DEPARTMENT OF MATHEMATICS
CITY COLLEGE OF CUNY
NEW YORK, NEW YORK 10031
E-MAIL: mmarcus@mail.idt.net

DEPARTMENT OF MATHEMATICS
COLLEGE OF STATEN ISLAND, CUNY
STATEN ISLAND, NEW YORK 10314
E-MAIL: jrosen3@mail.idt.net