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Joint continuity and a Doob-Meyer type decomposition for renormalized intersection local times

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ABSTRACT. – We study renormalized self-intersection local times $\gamma_n(\mu_x;t)$ for Lévy processes, where the n-fold multiple points are weighted by the translate μ_x of an arbitrary measure μ. General sufficient conditions are provided which insure that $\gamma_n(\mu_x;t)$ is a.s. jointly continuous in x and t. Our proof involves a new Doob-Meyer type decomposition of $\gamma_n(\mu_x;t)$ as the difference of a martingale and a lower order renormalized self-intersection local time. © Elsevier, Paris

RÉSUMÉ. – Nous étudions pour certains processus de Lévy les temps locaux d'auto-intersection renormalisés $\gamma_n(\mu_x;t)$ définis en affectant aux points de multiplicité n un poids donné par la translatée μ_x une mesure μ. Nous donnons des conditions suffisantes pour la continuité de $\gamma_n(\mu_x;t)$ comme fonction du couple (x,t). La preuve utilise une décomposition de Doob-Meyer nouvelle de $\gamma_n(\mu_x;t)$ comme différence d'une martingale et d'un temps local d'intersection renormalisé d'ordre inférieur. © Elsevier, Paris

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1. INTRODUCTION

The object of this paper is to establish the almost sure joint continuity of renormalized intersection local times for multiple self-intersections of a large class of Lévy processes in R^d including the symmetric stable processes in the plane. This builds on our work in [14] which dealt with intersection local times weighted by Lebesgue measure, and on our work in [12] which dealt with continuity in the spatial variable only. Our main technical tools are the Isomorphism Theorem for renormalized intersection local times developed in [12] and a Doob-Meyer type decomposition for renormalized intersection local times in terms of a martingale and a lower order renormalized intersection local time.

Intersection local times "measure" the amount of self-intersections of a stochastic process, say, $X(t) \in R^d$. To define the n−fold intersection local time, the natural approach is to set

$$\alpha_{n,\epsilon}(\mu;t) \stackrel{def}{=} \int \int_{\{0 \leq t_1 \leq \cdots \leq t_n \leq t\}} f_\epsilon(X(t_1) - x) \prod_{j=2}^{n} f_\epsilon(X(t_j) - X(t_{j-1}))\, dt_1 \cdots dt_n\, d\mu(x) \tag{1.1}$$

where f_ϵ is an approximate δ−function at zero, and take the limit as $\epsilon \to 0$. Intuitively, this gives a measure of the set of times $(t_1, \ldots, t_n)$ such that

$$X(t_1) = \cdots = X(t_n) = x,$$

where the "n-multiple points" $x \in R^d$ are weighted by the measure μ. However, in general, this limit does not exist because of the effect of the integral in the neighborhood of the diagonal. The method used to compensate for this is called renormalization. One subtracts from $\alpha_{n,\epsilon}(\mu, t)$ terms involving lower order intersections $\alpha_{k,\epsilon}(\mu, t)$ for $k < n$, in such a way that a finite limit results. This was originally done by Varadhan [15] for double intersections of Brownian motion in the plane with μ taken to be Lebesgue measure. Varadhan's work stimulated a large body of research which is summarized by Dynkin in [4]. Renormalized intersection local times have turned out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener and stable sausage in the plane and fluctuations of the range of stable random walks. (See Le Gall [7, 6], Le Gall-Rosen [9] and Rosen [13]). For a clear account of progress concerning Brownian intersection local times up to 1990 see Le Gall's lecture notes [8]. For more recent results see Bass and Khoshnevisan [1], Rosen [14] and Marcus and Rosen [12].

Let $(\Omega, \mathcal{F}(t), X(t), P^x)$ be a symmetric Lévy processes in R^d, $d = 1, 2$ with transition density $p_t(x,y) = p_t(x-y)$ and 1-potential density $u^1(x,y) = u^1(x-y)$. Since intersection local times are trivial for processes which have an actual local time we only consider Lévy processes for which $u^1(0) = \infty$. The results obtained in this paper are valid for a large class of radially symmetric Lévy processes which we say are in Class B. This class contains the symmetric stable processes and many processes in their domains of attraction. Class B is defined later in this section, see (1.15).

We will take the function f_ϵ appearing in (1.1) to be a smooth approximate identity. That is, $f_\epsilon(y)$ is a smooth positive symmetric function on $(y, \epsilon) \in R^d \times (0, 1]$ with support in the ball of radius ϵ and such that $\int f_\epsilon(y)\, dy = 1$.

For a Lévy processes $X(t) \in R^d$ and measure μ on R^d let

$$\gamma_{n,\epsilon}(\mu; t) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (u_\epsilon^1(0))^k \alpha_{n-k,\epsilon}(\mu; t) \tag{1.2}$$

where $u_\epsilon^1(0) = \int f_\epsilon(y) u^1(y)\, dy$. Heuristically, one may think of $\gamma_{n,\epsilon}(\mu; t)$ as

$$\begin{aligned} \gamma_{n,\epsilon}(\mu; t) = \int \int_{\{0 \le t_1 \le \cdots \le t_n \le t\}} & f_\epsilon(X(t_1) - x) \\ & \prod_{j=2}^{n} \left\{ f_\epsilon(X(t_j) - X(t_{j-1})) - \delta(t_j - t_{j-1}) u_\epsilon^1(0) \right\} \\ & dt_1 \cdots dt_n\, d\mu(x), \end{aligned} \tag{1.3}$$

where $\delta(\cdot)$ is the 'δ-function'. This formulation compensates for the difficulties caused when various of the t_i are close to each other. We set

$$\gamma_n(\mu; t) = \lim_{\epsilon \to 0} \gamma_{n,\epsilon}(\mu; t) \tag{1.4}$$

whenever the limit exists in $L^2(P^y)$, and refer to $\gamma_n(\mu; t)$ as a renormalized intersection local time. For any measure μ on R^d and $x \in R^d$, we let μ_x denote the translation of μ by x. The main goal of this paper is to study the a.s. continuity of $\{\gamma_n(\mu_x; t);\ (x,t) \in R^d \times R_+\}$.

Since we are dealing with processes which do not hit points, one cannot talk about the amount of 'multiple time' spent at a point x. However, if μ is concentrated near the origin, $\gamma_n(\mu_x; t)$ relates to the the 'n-multiple time' spent near x, and hence $\{\gamma_n(\mu_x; t);\ (x,t) \in R^d \times R_+\}$ can be thought of as an analogue of the local time process $\{L_t^x;\ (x,t) \in R^d \times R_+\}$.

We will obtain a sufficient condition for a.s. continuity of $\{\gamma_n(\mu_x; t);\ (x,t) \in R^d \times R_+\}$ in terms of a Gaussian chaos process which we now define. Known results about continuity of Gaussian chaos processes then allow us to provide concrete sufficient conditions for the a.s. continuity of $\{\gamma_n(\mu_x; t);\ (x,t) \in R^d \times R_+\}$.

We use $\mathcal{G}^m$ to denote the class of positive measures μ for which

$$\int\int (u^1(x,y))^m \, d\mu(x)\, d\mu(y) < \infty. \tag{1.5}$$

It should be understood that when we say $\mu \in \mathcal{G}^m$, that this is with respect to the 1-potential $u^1(x,y)$ of some given Lévy process.

Let $\{G_{x,\epsilon};\ (x,\epsilon) \in R^d \times (0,1]\}$ be the mean zero Gaussian process with covariance

$$E(G_{x,\epsilon} G_{x',\epsilon'}) = \int\int u^1(y-y') f_\epsilon(x-y) f_{\epsilon'}(x'-y')\, dy\, dy'.$$

It is easily checked that the right hand side is positive definite on $\left(R^d \times (0,1]\right)^2$. Now let $G_{(1),x,\epsilon}, \dots, G_{(2n),x,\epsilon}$ denote $2n$ independent copies of $G_{x,\epsilon}$.

For $(v_1, \dots, v_{2n}) \in (R^d)^{2n}$ and $\mu \in \mathcal{G}^{2n}$ define the decoupled Gaussian chaos $G_{2n,dec,\mu}(v_1, \dots, v_{2n})$ as the $\epsilon \to 0$ limit of

$$G_{2n,dec,\mu,\epsilon}(v_1, \dots, v_{2n}) = \int \prod_{j=1}^{2n} G_{(j),x+v_j,\epsilon}\, d\mu(x). \tag{1.6}$$

It is easily checked that this limit exists in $L^2(P_{G_{(1)},\dots,G_{(2n)}})$ where $P_{G_{(1)},\dots,G_{(2n)}}$ denotes probability with respect to $G_{(1)}, \dots, G_{(2n)}$. $G_{2n,dec,\mu}(v_1, \dots, v_{2n})$ has mean zero and for $\mu, \nu \in \mathcal{G}^{2n}$ we have

$$\begin{aligned} &E_{G_{(1)},\dots,G_{(2n)}}(G_{2n,dec,\mu}(v_1,\dots,v_{2n}) G_{2n,dec,\nu}(w_1,\dots,w_{2n})) \\ &\quad = \int\int \prod_{j=1}^{2n} u^1(x+v_j, y+w_j)\, d\mu(x)\, d\mu(y). \end{aligned} \tag{1.7}$$

For details see [11].

We can now state our main result on a.s. continuity of renormalized intersection local times.

THEOREM 1. – *Let* $\mu \in \mathcal{G}^{2n}$. *If* $\{G_{2n,dec,\mu}(v);\ v \in (R^d)^{2n}\}$ *is continuous a.s. then* $\{\gamma_n(\mu_x;t);\ (x,t) \in R^d \times R_+\}$ *is continuous a.s.*

With $v = (v_1, \ldots, v_{2n})$, $w = (w_1, \ldots, w_{2n})$ and $\mu \in \mathcal{G}^{2n}$, let us define the metric on $(R^d)^{2n}$

(1.8)

$$\begin{aligned} &\tau(v,w) \\ &\overset{def}{=} \left(E_{G_{(1)},\ldots,G_{(2n)}}(\{G_{2n,dec,\mu}(v_1,\ldots,v_{2n}) - G_{2n,dec,\mu}(w_1,\ldots,w_{2n})\}^2)\right)^{1/2} \\ &= \left(2\int \left((u^1(x,y))^{2n} - \prod_{j=1}^{2n} u^1(x+v_j, y+w_j)\right) d\mu(x)\, d\mu(y)\right)^{1/2}. \end{aligned}$$

A well known result about the a.s. continuity of Gaussian chaos processes allows us to state the following criteria for the a.s. continuity of renormalized intersection local times in terms of the metric $\tau(v,w)$ defined in (1.8). Let $N_\tau(B,\epsilon)$ be the minimum number of balls of radius ϵ, in the metric τ, that covers B, the unit ball in $(R^d)^{2n}$. $\log N_\tau(B,\cdot)$ is called the metric entropy of B with respect to τ.

THEOREM 2. – *Let* $\mu \in \mathcal{G}^{2n}$. *If*

$$\int_0^\infty (\log N_\tau(B,\epsilon))^n\, d\epsilon < \infty \tag{1.9}$$

then $\{\gamma_n(\mu_x;t);\ (x,t) \in R^d \times R_+\}$ *is continuous a.s.*

Theorem 4 of [11] describes simple concrete conditions which guarantee (1.9) and hence the a.s. continuity of $\{\gamma_n(\mu_x;t);\ (x,t) \in R^d \times R_+\}$.

An important technical tool for proving Theorem 1 and an interesting result in its own right is a Doob-Meyer type decomposition for renormalized intersection local times. If μ is a measure on R^d with bounded potential $U\mu$, and L_t^μ denotes the continuous additive functional with potential $E^x(L_\infty^\mu) = U\mu(x)$, it is easily seen using additivity and the Markov property that

$$E^x(L_\infty^\mu \mid \mathcal{F}_t) = L_t^\mu + U\mu(X(t)) \tag{1.10}$$

or equivalently $U\mu(X(t)) = M_t - L_t^\mu$ where $M_t = E^x(L_\infty^\mu \mid \mathcal{F}_t)$. This is the Doob-Meyer decomposition for the potential $U\mu(X(t))$. This decomposition has proven very useful in constructing and analyzing continuous additive functionals. We present an analogue of the Doob-Meyer decomposition related to the renormalized self-intersection local time $\gamma_n(\mu;t)$. We note that $\gamma_n(\mu;t)$ is not increasing in t. In fact, $\gamma_n(\mu;t)$ is defined so that $\int e^{-t}E^x(\gamma_n(\mu;t))\,dt=0$. (This can be seen by first replacing $\gamma_n(\mu;t)$ by $\gamma_{n,\epsilon}(\mu;t)$ and then passing to the limit).

Let $Y(t)$ denote our Lévy process $X(t)$ killed at an independent mean-1 exponential time λ. For our next theorem, the renormalized intersection local times $\gamma_k(\mu;t)$ will be defined for the process $Y(t)$ in place of $X(t)$, and we write $\gamma_k(\mu)$ in place of $\gamma_k(\mu;\infty)$.

Before presenting our general theorem, we first describe our results in the simplest case, $n=2$. Let π_t^μ be the random additive measure-valued process defined by

$$\pi_t^\mu(A)=L_t^{1_A\cdot\mu}=\int_0^t 1_A(Y(s))\,dL_s^\mu \tag{1.11}$$

for $A\subseteq R^d$. Our next theorem will imply the following analogue of the Doob-Meyer decomposition $U^1\mu(Y(t))=M_t-L_t^\mu$:

$$U^1\pi_t^\mu(Y(t))=\bar{M}_t-\gamma_2(\mu,t). \tag{1.12}$$

Here $\bar{M}_t$ is the martingale $E^x(\gamma_2(\mu)\,|\mathcal{F}_t)$ and $U^1\pi_t^\mu(Y(t))=\int_0^t u^1(Y(t)-Y(s))\,dL_s^\mu$ is the potential of the random measure π_t^μ.

We now describe the general case. We use $h_x(y)\stackrel{def}{=}h(y-x)$ for any function h. Let $u_v^1\cdot\mu$ denote, as usual, the measure obtained by multiplying $\mu(\,dx)$ by $u_v^1(x)$. We will see that under the conditions of Theorem 1, we can find an a.s. continuous version of $\{\gamma_{n-1}(u_v^1\cdot\mu;t)\,;\,(v,t)\in R^d\times R_+\}$. Using such a version we define

$$\gamma_{n-1}(u_{Y(t)}^1\cdot\mu;t)\stackrel{def}{=}\gamma_{n-1}(u_v^1\cdot\mu;t)\,|_{v=Y(t)}. \tag{1.13}$$

We also make the convention that $\gamma_0(\mu;t)=\alpha_0(\mu;t)=|\mu|$, the total mass of μ.

THEOREM 3. – *Under the conditions of Theorem 1, for each $t>0$ and $y\in R^d$*

$$E^y(\gamma_n(\mu)|\,\mathcal{F}_t)=\gamma_n(\mu;t)+\gamma_{n-1}(u_{Y(t)}^1\cdot\mu;t). \tag{1.14}$$

Let us point out that in the special case of Brownian motion in the plane, with μ taken to be Lebesgue measure, the decomposition (1.14) together with an exact description of the martingale as a stochastic integral was obtained by Bertoin [2] for $n = 2$ and by Bass and Khoshnevisan [1] for general n. Bertoin showed, in addition, that $\gamma_2(\mu, t)$ has zero quadratic variation. This guarantees that the decomposition is unique, and gives an intrinsic characterization of $\gamma_2(\mu, t)$ which does not involve any limiting proceedure. The extent to which this can be generalized to other processes, other measures and higher order renormalized intersection local times $\gamma_n(\mu, t)$ is still an open question.

* * *

We now describe the Lévy processes in Class B. We say that a strongly symmetric Lévy process belongs to Class B if it is radially symmetric, its Lévy exponent $\psi(\lambda)$ is regularly varying as $|\lambda| \to \infty$, its transition density $p_t(r)$ is bounded outside any neighborhood of the origin $r = 0$ uniformly in t, and, if $p_t^{(j)}(r)$ denotes the j'th derivative of $p_t(r)$ with respect to r, then for any $j = 1, \dots$ we can find $C_j < \infty$ such that

$$|p_t^{(j)}(2r)| \le C_j p_t(r)/r^j \tag{1.15}$$

for all t, r. In addition, we assume that $\int |x|^\beta u^1(x)\, dx < \infty$ for some $\beta > 0$. For this, it suffices that $\int |x|^\beta p_t(x)\, dx < \infty$ for some $t > 0$, and in particular this last condition is satisfied if $X(t)$ is in the domain of attraction of some stable process. It is readily checked that Class B is contained in the Class A of [12]. It is also easily checked that (1.15) holds for Brownian motion, and hence, using subordination, for the symmetric stable processes. We caution the reader that our Class A and Class B differ from those which appear in earlier classifications of Lévy processes, .e.g. [5].

Assume now that $\mu \in \mathcal{G}^{2n}$ and define $f_{\delta,x}(y) := f_\delta(y - x)$. The Cauchy-Schwarz inequality says that for any $a, b \in R^d$

$$\int\int \left(u^1(x,y)\right)^{2n} d\mu_a(x)\, d\mu_b(y) \le \int\int \left(u^1(x,y)\right)^{2n} d\mu(x)\, d\mu(y) \tag{1.16}$$

so that

$$\int\int \left(u^1(x,y)\right)^{2n} d\mu^\delta(x)\, d\mu^\delta(y) \le \int\int \left(u^1(x,y)\right)^{2n} d\mu(x)\, d\mu(y) \tag{1.17}$$

where $\mu^\delta = \mu * f_\delta = \mu(f_{\delta,x})\, dx$. (Here we use the notation $\mu(f_{\delta,x}) = \int f_{\delta,x}(y)\, d\mu(y)$). It is easy to check that for each $\delta > 0$, μ^δ has a bounded density. Since $\int u^1(x)\, dx = 1$ and by assumption $u^1(x)$ is bounded outside any neighborhood of the origin, we have that $(u^1(x))^{2n}$ is integrable outside any neighborhood of the origin, and consequently (1.17) implies that $u^1(x) \in L^{2n}(R^d, dx)$. The following Lemma will then follow from a straightforward modification of [14].

LEMMA 1. – *Let* $\mu \in \mathcal{G}^{2n}$. *Then for any* $0 \le k \le n$

$$\left\{\gamma_k\left(\prod_{j=1}^{n-k} u^1_{v_j} \cdot \mu^\delta_x; t\right);\ (x, v, t, \delta) \in R^d \times (R^d)^{n-k} \times R_+ \times (0, 1]\right\}$$

is continuous a.s.

The only point which requires verification is that $u^1(x)$ satisfies the conditions of [14]. The basic condition of [14] is that $u^1(x) \in L^{2n-1}(R^d, dx)$. The fact that in our case we actually have $u^1(x) \in L^{2n}(R^d, dx)$ gives us plenty of 'room to maneuver' and allows us to readily check all the other conditions of [14].

Lemma 1 does not say that the continuity is uniform in δ, a fact which would basically imply the conclusion of Theorem 1. An important ingredient in our proof of Theorem 1 will be to show that the continuity in Lemma 1 is actually uniform in δ.

2. JOINT CONTINUITY

Proof of Theorem 1. – We will prove this theorem when, in the definition of $\gamma_n(\mu; t)$, (1.1) and (1.2), our Lévy process $X(t)$ has been replaced by $Y(t)$, which is $X(t)$ killed at an independent mean-1 exponential time λ. Fubini's theorem will then yield our theorem for $X(t)$.

Set $\mu^\delta = \mu * f_\delta$. Our proof consists of two steps. We first show that $\gamma_n(\mu^\delta_x; t)$ converges a.s. locally uniformly in $(x, t) \in R^d \times R_+$ as $\delta \to 0$, and then we identify the limit, which is a.s. continuous by Lemma 1, with $\gamma_n(\mu_x; t)$.

We begin with an overview of our first step. For facts about the Wick power chaos $: G^{2n} : (\mu)$ see [12]. Under the conditions of our Theorem it follows from Theorem 3 of [11] that the Wick power chaos process $\{: G^{2n} : (\mu_x);\ x \in R^d\}$ is continuous a.s. Hence, a.s., $\int : G^{2n} : (\mu_y) f_\delta(x - y)\, dy$ converges to $: G^{2n} : (\mu_x)$ locally uniformly in $x \in R^d$ as $\delta \to 0$. It will follow from the Isomorphism Theorem, Theorem 4.1 of [12], that a.s. $\gamma_n(\mu^\delta_x) \to \gamma_n(\mu_x)$ locally uniformly in $x \in R^d$ as $\delta \to 0$, (see (3.6)-(3.7)). Using martingale techniques we will be able to show that $E^y(\gamma_n(\mu^\delta_x) | \mathcal{F}_t)$ converges a.s. locally uniformly in

$(x,t) \in R^d \times R_+$ as $\delta \to 0$ for each $y \in R^d$. If, instead of $E^y(\gamma_n(\mu_x^\delta)|\,\mathcal{F}_t)$, we could show that $\gamma_n(\mu_x^\delta;t)$ converges a.s., the proof of our first step would then be complete. Although $E^y(\gamma_n(\mu_x^\delta)|\,\mathcal{F}_t)$ is not the same as $\gamma_n(\mu_x^\delta;t)$, they differ by a term of lower order, as shown by the following Lemma, and we will be able to complete our argument by induction. The following Lemma, which is proven in section 6, is a special case of Theorem 3.

LEMMA 2. – *Let $\mu \in \mathcal{G}^{2n}$. Then for any $0 \le k \le n$, $t > 0$, $\delta > 0$, and $y \in R^d$*

$$E^y\left(\gamma_k\left(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta\right)|\,\mathcal{F}_t\right) = \gamma_k\left(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta; t\right) + \gamma_{k-1}\left(u_{Y(t)}^1 \prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta; t\right).$$

We can now give the details of our proof that $\gamma_n(\mu_x^\delta;t)$ converges a.s. locally uniformly in $(x,t) \in R^d \times R_+$ as $\delta \to 0$. If $\mathcal{S}$ is a subset of Euclidean space we will say that $\{Z_\epsilon(x);\ (\epsilon,x) \in (0,1] \times \mathcal{S}\}$ converges rationally locally uniformly on $\mathcal{S}$ as $\epsilon \to 0$ if for any compact $K \in \mathcal{S}$, $Z_\epsilon(x)$ converges uniformly in $x \in K$ as $\epsilon \to 0$ when restricted to dyadic rational x, ϵ. The following Lemma is proven in section 3.

LEMMA 3. – *Let $\mu \in \mathcal{G}^{2n}$. If $\{G_{2n,dec,\mu}(v);\ v \in (R^d)^{2n}\}$ is continuous a.s. then for each $0 \le k \le n$,*

$$M_k(\mu_x^\delta, v, t) \equiv E^y\left(\gamma_k\left(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta\right)|\,\mathcal{F}_t\right)$$

converges a.s. rationally locally uniformly in $(x,v,t) \in R^d \times (R^d)^{n-k} \times R_+$ as $\delta \to 0$.

We claim that $\gamma_k(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta;t)$ converges a.s. locally uniformly in $(x,v,t,\delta) \in R^d \times (R^d)^{n-k} \times R_+$ as $\delta \to 0$ for each $0 \le k \le n$. The case $k = n$ is precisely our assertion that $\gamma_n(\mu_x^\delta;t)$ converges a.s. locally uniformly in $(x,t) \in R^d \times R_+$ as $\delta \to 0$. We argue inductively on k. Assume that our claim has been proven with k replaced by $k-1$. Then by Lemmas 2 and 3 we have that $\gamma_k(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta;t)$ converges a.s. rationally locally uniformly in $(x,v,t,\delta) \in R^d \times (R^d)^{n-k} \times R_+$ as $\delta \to 0$ for each $0 \le k \le n$, and Lemma 1 now allows us the remove the qualification 'rationally'. Our induction will be completed once we prove the $k = 0$ case, which is the $k = 0$ case of the following Lemma. This Lemma, which is proven in section 4, will be used again, at that time for all values of k, in the proof of Lemma 3. We note the convention that : G^0 : $(\mu) = |\mu|$, the total mass of μ.

LEMMA 4. – *Let* $\mu \in \mathcal{G}^{2n}$. *If* $\{G_{2n,dec,\mu}(v);\ v \in (R^d)^{2n}\}$ *is continuous a.s. then* $\{: G^{2k} : (\prod_{j=1}^{n-k} u^1_{v_j} \cdot \mu_x);\ (x,v) \in R^d \times (R^d)^{n-k}\}$ *is continuous a.s. for each* $0 \leq k \leq n$. *Furthermore,*

$$\sup_y \int \left(u^1(x,y)\right)^n d\mu(x) < \infty. \tag{2.1}$$

The proof of our Theorem will then be completed by the following Lemma which is proven in section 5.

LEMMA 5. – *Let* $\mu \in \mathcal{G}^{2n}$. *If* $\{G_{2n,dec,\mu}(v);\ v \in (R^d)^{2n}\}$ *is continuous a.s. then*

$$\gamma_n(\mu_x; t) = \lim_{\delta \to 0} \gamma_n(\mu^\delta_x; t)$$

in $L^2(P^z)$ *for any* $z \in R^d$.

3. MARTINGALE CONVERGENCE

Proof of Lemma 3.

We will use Lemma 4. To use this lemma we claim that in L^2, for each $0 \leq k \leq n$,

$$: G^{2k} : \left(\prod_{j=1}^{n-k} u^1_{v_j} \cdot \mu^\delta_x\right) = \int : G^{2k} : \left(\prod_{j=1}^{n-k} u^1_{v_j} \cdot \mu_y\right) f_{\delta,x}(y)\, dy \tag{3.1}$$

The simple argument needed to establish our claim will be used repeatedly below, so for ease of exposition we isolate it as the following lemma.

LEMMA 6. – *Let* $\{Z(x,\epsilon,\delta)\,;\ (x,\epsilon,\delta) \in R^d \times [0,1]^2\}$ *be a stochastic process such that*

(i) $Z(x,\epsilon,\delta) = \int Z(y,\epsilon,0) f_{\delta,x}(y)\, dy$ *in* L^2 *for each* $(x,\delta) \in R^d \times [0,1]$ *and* $\epsilon > 0$.
If

(ii) $\lim_{\epsilon \to 0} Z(x,\epsilon,\delta) = Z(x,0,\delta)$ *in* L^2 *uniformly in* $x \in R^d$ *for each* $\delta \in [0,1]$ *then*

$$Z(x,0,\delta) = \int Z(y,0,0) f_{\delta,x}(y)\, dy \qquad a.s. \tag{3.2}$$

If in addition

(iii) $\{Z(x,0,\delta)\,;\, x \in R^d\}$ *is a.s. continuous for each* $\delta \in [0,1]$

then, a.s.,

$$\{Z(x,0,\delta)\,;\, x \in R^d\} = \left\{\int Z(y,0,0) f_{\delta,x}(y)\,dy\,;\, x \in R^d\right\}. \tag{3.3}$$

To prove this lemma simply apply (ii) to (i) to obtain (3.2) for each $x \in R^d$ seperately, and then (3.3) follows from (iii).

To establish our claim (3.1) we apply Lemma 6 to

$$Z(x,\epsilon,\delta) =: G_\epsilon^{2k} : \left(\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta\right).$$

Here $: G_\epsilon^{2k} : (\mu) = \int : G_{y,\epsilon}^{2k} : d\mu(y)$. Condition (i) follows from the definitions and (ii) follows from the fact that $\{\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x^\delta\,;\, x \in R^d\}$ is bounded in $\mathcal{G}^{2k}$ for all $\delta \in [0,1]$. To see this we use the multiple Holder inequality

(3.4)

$$\begin{aligned}
&\int\int (u^1(y-y'))^{2k} \prod_{j=1}^{n-k} u_{v_j}^1(y) u_{v_j}^1(y')\,d\mu_x^\delta(y)\,d\mu_x^\delta(y') \\
&\le \left\{\int\int (u^1(y-y'))^{2n}\,d\mu_x^\delta(y)\,d\mu_x^\delta(y')\right\}^{k/n} \\
&\qquad \prod_{j=1}^{n-k}\left\{\int\int (u_{v_j}^1(y)u_{v_j}^1(y'))^n\,d\mu_x^\delta(y)\,d\mu_x^\delta(y')\right\}^{1/n} \\
&=\left\{\int\int (u^1(y-y'))^{2n}\,d\mu^\delta(y)\,d\mu^\delta(y')\right\}^{k/n} \prod_{j=1}^{n-k}\left\{\int (u_{v_j}^1(y))^n\,d\mu_x^\delta(y)\right\}^{2/n}
\end{aligned}$$

and then use (1.17) and (2.1) to see that this is bounded uniformly in all parameters. This proves (3.1).

Lemma 4 shows that $\int : G^{2k} : (\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_y) f_{\delta,x}(y)\,dy$ converges a.s. to $: G^{2k} : (\prod_{j=1}^{n-k} u_{v_j}^1 \cdot \mu_x)$ as $\delta \to 0$ locally uniformly in $(x,v) \in (R^d)^{n-k+1}$. Let $B(m) \subseteq (R^d)^{1+n-k}$ denote the ball of radius m and let $D \subseteq (R^d)^{1+n-k}$, $\bar{D} \subseteq \times(0,1]^2$ be countable dense subsets.

Using (3.1) we have that for any $m, \eta > 0$ we can find $\epsilon > 0$ such that

(3.5)
$$E_G\Big(\sup_{\substack{(x,v)\in B(m)\cap D\\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} \Big| : G^{2k} : \Big(\prod_{j=1}^{n-k} u^1_{v_j}\cdot \mu^\delta_x\Big) - : G^{2k} : \Big(\prod_{j=1}^{n-k} u^1_{v_j}\cdot \mu^{\delta'}_x\Big)\Big|\Big) \leq \eta.$$

Let $\rho(dx) = u^1(x)\,dx$ which is easily seen to be in $\mathcal{G}^1$ using $u^1(x) \in L^2(dx)$. We will use the abbreviation $\phi_v = \prod_{j=1}^{n-k} u^1_{v_j}$. Recalling from Lemma 1 the a.s continuity of $\gamma_k(\phi_v\cdot\mu^\delta_x)$, and using the notation of the Isomorphism Theorem, Theorem 4.1 of [12], we have that

$$\begin{aligned}
(3.6)\ & k!\,E^\rho\left(\sup_{\substack{(x,v)\in B(m)\\ (\delta,\delta')\in(0,\epsilon]}} |\gamma_k(\phi_v\cdot\mu^\delta_x) - \gamma_k(\phi_v\cdot\mu^{\delta'}_x)|\right)\\
&= k!\,E^\rho\left(\sup_{\substack{(x,v)\in B(m)\cap D\\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} |\gamma_k(\phi_v\cdot\mu^\delta_x) - \gamma_k(\phi_v\cdot\mu^{\delta'}_x)|\right)\\
&= E^\rho\left(\sup_{\substack{(x,v)\in B(m)\cap D\\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} |\mathcal{L}_k(\phi_v\cdot\mu^\delta_x) - \mathcal{L}_k(\phi_v\cdot\mu^{\delta'}_x)|\right)\\
&= E^\rho\Big(\sup_{\substack{(x,v)\in B(m)\cap D\\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} \Big| \sum_{i=0}^k \binom{k}{i}\frac{1}{2^{k-i}}\\
&\qquad E_G\Big\{(\mathcal{L}_i\times : G^{2(k-i)} :)(\phi_v\cdot\mu^\delta_x) - (\mathcal{L}_i\times : G^{2(k-i)} :)(\phi_v\cdot\mu^{\delta'}_x)\Big\}\Big|\Big)\\
&\leq E_G E^\rho\Big(\sup_{\substack{(x,v)\in B(m)\cap D\\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} \Big| \sum_{i=0}^k \binom{k}{i}\frac{1}{2^{k-i}}\\
&\qquad \Big\{(\mathcal{L}_i\times : G^{2(k-i)} :)(\phi_v\cdot\mu^\delta_x) - (\mathcal{L}_i\times : G^{2(k-i)} :)(\phi_v\cdot\mu^{\delta'}_x)\Big\}\Big|\Big).
\end{aligned}$$

In the third equality we used the fact that $E_G\big\{(\mathcal{L}_i\times : G^{2(k-i)} :)(\phi_v\cdot\mu^\delta_x)\big\} = 0$ for all $0 \leq i < k$.

It is easy to check that we can choose $\kappa > 0$ such that, with $f(x) = |x|^{-\kappa}$ on R^d, we have $f(x)\cdot dx \in \mathcal{G}^1$. Then, using our hypothesis that $\int |x|^\beta u^1(x)\,dx < \infty$ for some $\beta > 0$, we can choose $p \geq 1$ sufficiently large, such that with $1/q = 1 - 1/p$ we have

$$\begin{aligned}
&E^{\rho}(\{f(X(\lambda)\}^{-q/p}) \\
&= E^{\rho}(|X(\lambda)|^{\kappa q/p}) \\
&= \int\int |x+y|^{\kappa q/p} u^1(x)\,dx\,d\rho(y) \\
&= \int\int |x+y|^{\kappa q/p} u^1(x) u^1(y)\,dx\,dy < \infty.
\end{aligned}$$

Hence, using Holder's inequality and then the Isomorphism Theorem, Theorem 4.1 of [12] and (3.5), we see that (3.6)

(3.7)

$$\begin{aligned}
&\leq c\Bigg\{ E_G E^{\rho}(\sup_{\substack{(x,v)\in B(m)\cap D \\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} |\sum_{i=0}^{k} \binom{k}{i} \frac{1}{2^{k-i}} \\
&\quad \Big\{ (\mathcal{L}_i \times : G^{2(k-i)} :)(\phi_v \cdot \mu_x^{\delta}) - (\mathcal{L}_i \times : G^{2(k-i)} :)(\phi_v \cdot \mu_x^{\delta'}) \Big\} |^p f(X(\lambda))) \Bigg\}^{1/p} \\
&= c\frac{1}{2^k}\Bigg\{ E_G(\sup_{\substack{(x,v)\in B(m)\cap D \\ (\delta,\delta')\in(0,\epsilon]\cap\bar{D}}} | : G^{2k} : (\phi_v \cdot \mu_x^{\delta}) \\
&\qquad\qquad - : G^{2k} : (\phi_v \cdot \mu_x^{\delta'}) |^p G_{\rho} G_{f \cdot dx}) \Bigg\}^{1/p} \\
&\leq c\eta.
\end{aligned}$$

In the last step we used the equivalence of moments for a Gaussian chaos, see [10], section 3.2. We note that the Isomorphism Theorem, Theorem 4.1 of [12] is stated for ρ with compact support and f bounded, uniformly continuous and integrable, but the extension to our ρ, f is immediate.

Fix $y \in B_d(m/4)$, the ball of radius $m/4$ in R^d. For any $y' \in B_d(m/4)$ we have

(3.8)

$$\begin{aligned}
&E^{y}\left(\sup_{\substack{(x,v)\in B(m/4) \\ 0<\delta,\delta'\leq\epsilon}} |\gamma_k(\phi_v \cdot \mu_x^{\delta}) - \gamma_k(\phi_v \cdot \mu_x^{\delta'})| \right) \\
&= E^{y'}\left(\sup_{\substack{(x,v)\in B(m/4) \\ 0<\delta,\delta'\leq\epsilon}} |\gamma_k(\phi_{v+y-y'} \cdot \mu_{x+y-y'}^{\delta}) - \gamma_k(\phi_{v+y-y'} \cdot \mu_{x+y-y'}^{\delta'})| \right)
\end{aligned}$$

$$\leq E^{y'}\left(\sup_{\substack{(x,v)\in B(m)\\ 0<\delta,\delta'\leq\epsilon}} |\gamma_k(\phi_v\cdot\mu_x^\delta)-\gamma_k(\phi_v\cdot\mu_x^{\delta'})|\right).$$

Using this, (3.6) and (3.7) we see that for any $y \in B_d(m/4)$

$$\begin{aligned}(3.9)\qquad & E^y\left(\sup_{\substack{(x,v)\in B(m/4)\\ 0<\delta,\delta'\leq\epsilon}} |\gamma_k(\phi_v\cdot\mu_x^\delta)-\gamma_k(\phi_v\cdot\mu_x^{\delta'})|\right)\\ & \leq cE^\rho\left(\sup_{\substack{(x,v)\in B(m)\\ 0<\delta,\delta'\leq\epsilon}} |\gamma_k(\phi_v\cdot\mu_x^\delta)-\gamma_k(\phi_v\cdot\mu_x^{\delta'})|\right)\\ & \leq c\eta.\end{aligned}$$

This says that for any $y \in B_d(m/4)$

$$(3.10)\qquad E^y\left(\sup_{\substack{(x,v)\in B(m/4)\\ 0<\delta,\delta'\leq\epsilon}} \left\{M_k(\mu_x^\delta,v,\infty)-M_k(\mu_x^{\delta'},v,\infty)\right\}\right)\leq c\eta.$$

Since for any finite sets $F\subseteq B(m/4)$ and $F_\epsilon\subseteq(0,\epsilon]$ we have that

$$(3.11)\qquad \sup_{\substack{(x,v)\in F\\ \delta,\delta'\in F_\epsilon}} \left\{M_k(\mu_x^\delta,v,t)-M_k(\mu_x^{\delta'},v,t)\right\}$$

is a right continuous submartingale in t, we have from (3.10) that

$$(3.12)\qquad E^y\left(\sup_{t\geq 0}\sup_{\substack{(x,v)\in F\\ \delta,\delta'\in F_\epsilon}} \left\{M_k(\mu_x^\delta,v,t)-M_k(\mu_x^{\delta'},v,t)\right\}\right)\leq c\gamma.$$

with c independent of the finite sets F, F_ϵ. By monotone convergence this holds also if we take F, F_ϵ to be all rational elements in $B(m/4), (0,\epsilon]$ respectively. This proves that for each $0\leq k\leq n$ we have that $M_k(\mu_x^\delta,v,t)$ converges a.s. rationally locally uniformly in $(x,v,t)\in R^d\times(R^d)^{n-k}\times R_+$ as $\delta\to 0$ and completes the proof of Lemma 3.

4. CONTINUITY OF CHAOSES

Proof of Lemma 4. – Let us first show that $\{: G^{2k} : (\prod_{j=1}^{n-k} u^1_{v_j} \cdot \mu_x);\ (x,v) \in R^d \times (R^d)^{n-k}\}$ is a.s. locally bounded for each $0 \le k \le n$.

For any $0 \le k \le n$ set

(4.1)

$$(G_{2k,dec} \times : G^2 :_{n-k,dec})_\mu(w_1,\dots,w_{2k}; v_1,\dots,v_{n-k}) = \lim_{\epsilon\to 0} \int \prod_{i=1}^{2k} G_{(i),x+w_i,\epsilon} \prod_{j=1}^{n-k} \left\{ G^2_{(2k+j),x+v_j,\epsilon} - E(G^2_{(2k+j),x+v_j,\epsilon}) \right\} d\mu(x)$$

and

$$(G_{2k,dec} \times L_{n-k,dec})_\mu(w_1,\dots,w_{2k}; v_1,\dots,v_{n-k}) = \lim_{\epsilon\to 0} \int \prod_{i=1}^{2k} G_{(i),x+w_i,\epsilon} \prod_{j=1}^{n-k} L^{\rho_{\epsilon,x+v_j}}_{(j)}\, d\mu(x) \tag{4.2}$$

where $L^\nu_{(j)} = L^\nu_{(j),\infty}$, $j = 1,\dots,n-k$, denote the (total) CAF's with Revuz measure ν built up from independent copies $Y_{(j)}(\cdot)$ of $Y(\cdot)$. We will often write $w = (w_1,\dots,w_{2k}), v = (v_1,\dots,v_{n-k})$. It is easy to check that the limit in (4.1) exists in $L^2(dP_G)$, and by induction on $0 \le k \le n$, using the Isomorphism Theorem, see Theorem 4.3 of [12], it is easy to show that the limit in (4.2) exists in $L^2(dP_G \times dP^{\rho,\dots,\rho})$, uniformly in $(w,v) \in (R^d)^{n+k}$, for each $\mu \in \mathcal{G}^{2n}$. Here, $P^{\rho,\dots,\rho}$ is the product measure for the processes $Y_{(1)}(\cdot),\dots,Y_{(n-k)}(\cdot)$, each with initial measure $\rho(dx) = u^1(x)\,dx$. By the assumption of our Lemma and Theorem 5 of [11] we have that $\{(G_{2k,dec} \times : G^2 :_{n-k,dec})_\mu(w,v);\ (w,v) \in (R^d)^{n+k}\}$ is continuous a.s., so that an easy application of the Isomorphism Theorem, using the ideas of the previous section, shows that $\{(G_{2k,dec} \times L_{n-k,dec})_\mu(w,v);\ (w,v) \in (R^d)^{n+k}\}$ is continuous a.s. and that for any m

$$\| \sup_{(w,v)\in B(m)} |(G_{2k,dec} \times L_{n-k,dec})_\mu(w,v)| \, \|_{2,G\times L} \le c \| \sup_{(w,v)\in B(m)} |(G_{2k,dec} \times : G^2 :_{n-k,dec})_\mu(w,v)| \, \|_{2,G} < \infty \tag{4.3}$$

where $B(m)$ now denotes the ball of radius m in $(R^d)^{n+k}$ and $\|\cdot\|_{2,G}$, $\|\cdot\|_{2,G\times L}$ denote respectively the norms for $L^2(dP_G)$ and $L^2(dP_G \times dP^{\rho,\dots,\rho})$.

Thus, if we set

$$F_\delta(w,v) = \prod_{i=1}^{2k} f_\delta(w_i) \prod_{j=1}^{n-k} f_\delta(v_j)$$

we see that for each fixed $\delta > 0$,

$$\begin{aligned} (4.4)\quad & (G_{2k,dec} \times L_{n-k,dec})_\mu * F_\delta(w,v) \\ & \stackrel{def}{=} \int (G_{2k,dec} \times L_{n-k,dec})_\mu(w',v') F_\delta(w-w',v-v)\, dv'\, dw' \end{aligned}$$

is continuous a.s. and, for δ sufficiently small,

$$\begin{aligned} (4.5)\quad & \| \sup_{(w,v)\in B(m/2)} |(G_{2k,dec} \times L_{n-k,dec})_\mu * F_\delta(w,v)| \|_{2,G\times L} \\ & \le \| \sup_{(w,v)\in B(m)} |(G_{2k,dec} \times L_{n-k,dec})_\mu(w,v)| \|_{2,G\times L}. \end{aligned}$$

Furthermore, since the L^2 limit in (4.2) is uniform in $(w,v) \in (R^d)^{n+k}$, we have from Lemma 6 applied to

$$\begin{aligned} (4.6)\quad & Z((w,v),\epsilon,\delta) \\ & = \int \left(\prod_{i=1}^{2k} \left\{ \int G_{(i),x+w'_i,\epsilon} f_{\delta,x+w_i}(w'_i)\, dw'_i \right\} \right. \\ & \left. \prod_{j=1}^{n-k} \left\{ \int L_{(j)}^{\rho_{\epsilon,v'_j}} f_{\delta,x+v_j}(v'_j)\, dv'_j \right\} \right) d\mu(x) \end{aligned}$$

that

$$\begin{aligned} (4.7)\quad & \{(G_{2k,dec} \times L_{n-k,dec})_\mu * F_\delta(w,v)\,;\,(w,v)\in (R^d)^{n+k}\} \\ & = \left\{ \int \prod_{i=1}^{2k} G_{(i),x+w_i,\delta} \prod_{j=1}^{n-k} L_{(j)}^{\rho_{\delta,x+v_j}}\, d\mu(x)\,;\,(w,v)\in (R^d)^{n+k} \right\} \end{aligned}$$

Here we used the fact that both sides are a.s. continuous in $(w,v) \in (R^d)^{n+k}$ which follows easily since we are assuming that f_δ is smooth.

Since

$$E\Big(\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}L_{(j)}^{\rho_{\delta,x+v_j}}\,d\mu(x)\Big)\tag{4.8}$$
$$=\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}U^1f_\delta(x+v_j)\,d\mu(x),$$

using translation invariance as in the previous section we have that

(4.9)

$$\Big\|\sup_{(w,v)\in B(m/8)}\Big|\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}U^1f_\delta(x+v_j)\,d\mu(x)\Big|\Big\|_{2,G}$$
$$=\Big\|\sup_{(w,v)\in B(m/8)}\Big|E\Big(\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}L_{(j)}^{\rho_{\delta,x+v_j}}\,d\mu(x)\Big)\Big|\Big\|_{2,G}$$
$$\le\Big\|\sup_{|w|\le m/8}E\Big(\sup_{|v|\le m/8}\Big|\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}L_{(j)}^{\rho_{\delta,x+v_j}}\,d\mu(x)\Big|\Big)\Big\|_{2,G}$$
$$\le c\Big\|\sup_{|w|\le m/8}E^{\rho,\dots,\rho}\Big(\sup_{|v|\le m/4}\Big|\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}L_{(j)}^{\rho_{\delta,x+v_j}}\,d\mu(x)\Big|\Big)\Big\|_{2,G}$$
$$\le c\Big\|\sup_{(w,v)\in B(m/2)}\Big|\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}L_{(j)}^{\rho_{\delta,x+v_j}}\,d\mu(x)\Big|\Big\|_{2,G\times L}$$
$$=c\|\sup_{(w,v)\in B(m/2)}|(G_{2k,dec}\times L_{n-k,dec})_\mu*F_\delta(w,v)|\,\|_{2,G\times L}.$$

Hence by (4.5) we have that

$$\Big\|\sup_{(w,v)\in B(m/8)}\Big|\int\prod_{i=1}^{2k}G_{(i),x+w_i,\delta}\prod_{j=1}^{n-k}U^1f_\delta(x+v_j)\,d\mu(x)\Big|\Big\|_{2,G}\tag{4.10}$$
$$\le c\|\sup_{(w,v)\in B(m)}|(G_{2k,dec}\times L_{n-k,dec})_\mu(w,v)|\,\|_{2,G\times L}<\infty.$$

In the case of $k=0$ our last inequality together with (4.3) says that

$$\sup_{v\in B(m/8)}\int\prod_{j=1}^{n}U^1f_\delta(x+v_j)\,d\mu(x)\tag{4.11}$$
$$\le c\|\sup_{v\in B(m)}|(:G^2:_{n,dec})_\mu(v)|\,\|_{2,G}<\infty.$$

By assumption, $u^1(x)$ is continuous for $x \neq 0$, hence $\prod_{j=1}^n U^1 f_\delta(x+v_j) \to \prod_{j=1}^n u^1_{v_j}(x)$ as $\delta \to 0$ for each $x \notin \{v_1, \dots, v_n\}$. Since $\mu \in \mathcal{G}^{2n}$ is necessarily non-atomic, Fatou's Lemma applied to (4.11) now gives

(4.12)

$$\sup_{v \in B(m/8)} \int \prod_{j=1}^n u^1_{v_j}(x)\, d\mu(x) \leq c \| \sup_{v \in B(m)} |(: G^2 :_{n,dec})_\mu(v)| \, \|_{2,G} < \infty.$$

This shows that $: G^0 : (\prod_{j=1}^n u^1_{v_j} \cdot \mu_x) = \int \prod_{j=1}^n u^1_{v_j}(x)\, d\mu(x)$ is locally uniformly bounded, and continuity is proven similarly.

Using now the stationarity of G together with (4.12), we have that for any $y \in R^d$

$$\begin{aligned} \sup_{v \in B(m/8)} & \int \prod_{j=1}^n u^1_{v_j}(x)\, d\mu_y(x) \\ & \leq c \| \sup_{v \in B(m)} |(: G^2 :_{n,dec})_{\mu_y}(v)| \, \|_{2,G} \\ & = c \| \sup_{v \in B(m)} |(: G^2 :_{n,dec})_\mu(v)| \, \|_{2,G} < \infty. \end{aligned} \tag{4.13}$$

In particular this implies (2.1).

Now consider the case of $k > 1$. Let $\phi_v = \prod_{j=1}^{n-k} u^1_{v_j}$. As in (3.4) we have that $\phi_v \cdot \mu \in \mathcal{G}^{2k}$, and applying Lemma 6 to

$$\begin{aligned} & Z((w,v), \epsilon, \delta) \\ & = \int \left(\prod_{i=1}^{2k} \left\{ \int G_{(i), x+w_i', \epsilon} f_{\delta, x+w_i}(w_i')\, dw_i' \right\} \right. \\ & \qquad \left. \prod_{j=1}^{n-k} \left\{ \int U^1 f_\epsilon(v_j') f_{\delta, x+v_j}(v_j')\, dv_j' \right\} \right) d\mu(x) \end{aligned} \tag{4.14}$$

we see that

$$\begin{aligned} & G_{2k, dec, \phi_\cdot \cdot \mu} * F_\delta(w, v) \\ & \stackrel{def}{=} \int G_{2k, dec, \phi_{v'} \cdot \mu}(w') F_\delta(w - w', v - v')\, dw'\, dv' \\ & = \int \prod_{i=1}^{2k} G_{(i), x+w_i, \delta} \prod_{j=1}^{n-k} U^1 f_\delta(x + v_j)\, d\mu(x) \end{aligned} \tag{4.15}$$

in $L^2(dP_G)$ for each $(w,v) \in (R^d)^{n+k}$. Hence, for any finite $D \subseteq B(m/8) \subseteq (R^d)^{n+k}$, by (4.10) we have that

$$\text{(4.16)} \quad \begin{aligned} &\| \sup_{(w,v)\in D} |G_{2k,dec,\phi_\cdot \cdot \mu} * F_\delta(w,v)| \, \|_{2,G} \\ &\le \| \sup_{(w,v)\in B(m)} |(G_{2k,dec} \times L_{n-k,dec})_\mu(w,v)| \, \|_{2,G\times L}. \end{aligned}$$

As noted, $\phi_v \cdot \mu \in \mathcal{G}^{2k}$ and in fact, as a function of v, is continuous in $\mathcal{G}^{2k}$ as follows from the proof of Lemma 3.1 of [12]. Hence $\{G_{2k,dec,\phi_v\cdot\mu}(w)\,;\,(w,v) \in (R^d)^{n+k}\}$ is continuous in $L^2(dP_G)$. Therefore

$$\int G_{2k,dec,\phi_\cdot \cdot \mu} * F_\delta(w,v) \to G_{2k,dec,\phi_v\cdot\mu}(w)$$

in $L^2(dP_G)$ for fixed w,v, hence a.s for w,v in any finite $D \subseteq B(m/8)$. This shows that

$$\text{(4.17)} \quad \begin{aligned} &\| \sup_{(w,v)\in D} |G_{2k,dec,\phi_v \cdot \mu}(w)| \, \|_{2,G} \\ &\le \| \sup_{(w,v)\in B(m)} |(G_{2k,dec} \times L_{n-k,dec})_\mu(w,v)| \, \|_{2,G\times L} < \infty. \end{aligned}$$

By the Monotone Convergence Theorem this will also hold if we take D to be any countable dense set in $B(m/8)$. Since $\{G_{2k,dec,\phi_v\cdot\mu}(w)\,;\,(w,v) \in (R^d)^{n+k}\}$ is continuous in L^2, we can assume that we are dealing with a separable version. Because of separability we can replace D in (4.17) by $B(m/8)$. This shows that $G_{2k,dec,\phi_v\cdot\mu}(w)$ is bounded uniformly in $(w,v) \in B(m/8)$ a.s. Then, Theorem 3 of [11] shows that : $G^{2k} : (\phi_v \cdot \mu_x)$ is bounded uniformly in $(x,v) \in B(m/16)$ a.s. The result on continuity is treated similarly. This completes the proof of lemma 4.

5. L^2 CONVERGENCE

Proof of Lemma 5.

LEMMA 7. – *If $\mu \in \mathcal{G}^{2n}$ satisfies (2.1), then for each $t \in R_+$ and $y \in R^d$ the limit*

$$\gamma_n(\mu_y;t) = \lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu_y;t)$$

exists in $L^2(P^x)$, uniformly in $y \in R^d$, and $\{\gamma_n(\mu_y;t)\,;\,y \in R^d\}$ is continuous $L^2(P^x)$.

Using this Lemma we can apply Lemma 6 to $Z(x,\epsilon,\delta) = \gamma_{n,\epsilon}(\mu_x^\delta;t)$ to see that $\gamma_n(\mu_x^\delta;t) = \int \gamma_n(\mu_y;t) f_{\delta,x}(y)\,dy$ in $L^2(P^z)$. The continuity of $\{\gamma_n(\mu_y;t)\,;\, y\in R^d\}$ in $L^2(P^z)$ now implies that $\gamma_n(\mu_x^\delta;t)\to\gamma_n(\mu_x;t)$ in $L^2(P^z)$. This completes the proof of Lemma 5.

Proof of Lemma 7. – This proof is meant to be read in conjunction with [12], on which we rely for ideas and notation. Using $q_t(x)$ to denote transition density and semigroup of our killed process, we find the following analogue of (5.24) of [12]:

(5.1)

$$E^x(\alpha_{n_1,\epsilon_1}(\mu;t)\alpha_{n_2,\epsilon_2}(\nu;t)) = \sum_{s\in\mathcal{S}} \int\int_{\{\sum t_p\le t\}} \prod_{p=1}^{n_1+n_2} q_{t_p}(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}) \prod_p dt_p \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)$$

where $\mathcal{S}$ is the set of mappings

$$s:\{1,\ldots,n_1+n_2\}\mapsto\{1,2\}$$

such that $|s^{-1}(i)| = n_i$, $i=1,2$ and $c(p) = |\{m\le p|\, s(m)=s(p)\}|$, $1\le p\le n_1+n_2$. By convention we set $x_{s(0)} = x$ and $c(0)=0$.

Let us analyze the changes which occur in (5.1) when we replace one of the factors, say $\alpha_{n_i,\epsilon_1}(\mu,t)$, by $\binom{n_1-1}{k_1}(u^1_{\epsilon_1}(0))^{k_1}\alpha_{n_1-k_1,\epsilon}(\mu,t)$. As in [12] we see that

(5.2)

$$E^x(\binom{n_1-1}{k_1}(u^1_{\epsilon_1}(0))^{k_1}\alpha_{n_1-k_1,\epsilon_1}(\mu,t)\alpha_{n_2,\epsilon_2}(\nu,t)) = \sum_{\substack{D_1\subseteq\{2,\ldots,n_1\}\\ |D_1|=k_1}} \sum_{s\in\mathcal{S}_{D_1}} \int\int_{\{\sum_{(\widetilde{D}_1)^c} t_p\le t\}} \prod_{l\in D_1} u^1(y_{1,l}) \prod_{\substack{p=1\\ (s(p),c(p))\notin(1,D_1)}}^{n_1+n_2} q_{t_p}(x_{s(p)} + \sum_{\substack{j=1\\ (s(p),j)\notin(1,D_1)}}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{\substack{j=1\\ (s(p-1),j)\notin(1,D_1)}}^{c(p-1)} y_{s(p-1),j}) \prod_{p\in\widetilde{D}_1^c} dt_p \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)$$

where $\mathcal{S}_{D_1}$ is the subset of $\mathcal{S}$ such that $\widetilde{D}_1 = \{p \,|\, (s(p), c(p)) \in (1, D_1)\} \subseteq B_s$. Using the notational convention that $\delta(s)\,ds$ designates the probability measure which places unit mass at $s = 0$, we can rewrite this as

$$(5.3)\quad E^x\left(\binom{n_1-1}{k_1}(u^1_{\epsilon_1}(0))^{k_1}\alpha_{n_1-k_1,\epsilon_1}(\mu,t)\alpha_{n_2,\epsilon_2}(\nu,t)\right)$$

$$= \sum_{\substack{D_1\subseteq\{2,\dots,n_1\}\\ |D_1|=k_1}} \sum_{s\in\mathcal{S}_{D_1}} \int\int_{\{\sum t_p\le t\}} \prod_{\widetilde{D}_1} u^1(y_{s(p),c(p)})\delta(t_p)$$

$$\prod_{\substack{p=1\\ (s(p),c(p))\notin(1,D_1)}}^{n_1+n_2} q_{t_p}(x_{s(p)} + \sum_{\substack{j=1\\ (s(p),j)\notin(1,D_1)}}^{c(p)} y_{s(p),j}$$

$$- x_{s(p-1)} - \sum_{\substack{j=1\\ (s(p-1),j)\notin(1,D_1)}}^{c(p-1)} y_{s(p-1),j})$$

$$\prod_p dt_p \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2).$$

Arguing once more as in [12], this leads to

$$(5.4)$$

$$E^x(\gamma_{n,\epsilon_1}(\mu;t)\gamma_{n,\epsilon_2}(\nu;t))$$

$$= \sum_{s\in\mathcal{S}} \int\int_{\{\sum t_p\le t\}} \prod_{p\in B_s} \{q_{t_p}(y_{s(p),c(p)}) - u^1(y_{s(p),c(p)})\delta(t_p)R_{y_{s(p),c(p)}}\}$$

$$\prod_{p\in B_s^c} q_{t_p}(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j})$$

$$\prod_p dt_p \prod_{i=1}^{2}\prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)$$

where R_y is the operator which acts on a function of the variable y by setting y equal to 0. If now $\mathcal{D}_y = I - R_y$ where I denotes the identity operator, we can write our last equation as

(5.5)

$$
\begin{aligned}
&E^x(\gamma_{n,\epsilon_1}(\mu;t)\gamma_{n,\epsilon_2}(\nu;t))\\
&=\sum_{s\in\mathcal{S}}\int\int_{\{\sum t_p\le t\}}\prod_{p\in B_s}\Big(\{q_{t_p}(y_{s(p),c(p)})-u^1(y_{s(p),c(p)})\delta(t_p)\}\\
&\qquad\qquad+u^1(y_{s(p),c(p)})\delta(t_p)\mathcal{D}_{y_{s(p),c(p)}}\Big)\\
&\qquad\prod_{p\in B_s^c} q_{t_p}(x_{s(p)}+\sum_{j=1}^{c(p)} y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)} y_{s(p-1),j})\\
&\qquad\prod_p dt_p\prod_{i=1}^{2}\prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)\\
&=\sum_{s\in\mathcal{S}}\sum_{A\subseteq B_s}\int\int_{\{\sum t_p\le t\}}\prod_{p\in A}\{q_{t_p}(y_{s(p),c(p)})-u^1(y_{s(p),c(p)})\delta(t_p)\}\\
&\qquad\prod_{p\in B_s-A} u^1(y_{s(p),c(p)})\delta(t_p)\mathcal{D}_{y_{s(p),c(p)}}\\
&\qquad\prod_{p\in B_s^c} q_{t_p}(x_{s(p)}+\sum_{j=1}^{c(p)} y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)} y_{s(p-1),j})\\
&\qquad\prod_p dt_p\prod_{i=1}^{2}\prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)\\
&=\sum_{s\in\mathcal{S}}\sum_{A\subseteq B_s}\int\int_{\{\sum_{B_s^c} t_p\le t\}}\\
&\left(\int_{\{\sum_A t_p\le t-\sum_{B_s^c} t_p\}}\prod_{p\in A}\{q_{t_p}(y_{s(p),c(p)})-u^1(y_{s(p),c(p)})\delta(t_p)\}\,dt_p\right)\\
&\qquad\prod_{p\in B_s-A} u^1(y_{s(p),c(p)})\mathcal{D}_{y_{s(p),c(p)}}\prod_{p\in B_s^c} q_{t_p}(x_{s(p)}\\
&\qquad+\sum_{j=1}^{c(p)} y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)} y_{s(p-1),j})\\
&\qquad\prod_{B_s^c} dt_p\prod_{i=1}^{2}\prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2).
\end{aligned}
$$

When we compare this with the analysis in [12] we see that the main new element comes from the inner integral involving A. In order to handle this, we now show

LEMMA 8

$$|\int_{\{\sum_A t_p \leq T\}} \prod_{p\in A} \{q_{t_p}(y_{s(p),c(p)}) - u^1(y_{s(p),c(p)})\delta(t_p)\}\, dt_p| \leq C\left(\int_T^\infty q_s(0)\, ds\right)^{|A|} \tag{5.6}$$

Proof of Lemma 8. – Set $Q_T = \int_T^\infty q_s(0)\, ds$. We note that Q_T is regularly varying at $T = 0$ (we include the possibility that Q_T is slowly varying) and $Q_T \nearrow u^1(0) = \infty$ as $T \searrow 0$. Let us analyze

(5.7)

$$\begin{aligned}
&\int_{\{\sum_{j=1}^k t_j \leq T\}} \prod_{j=1}^k \{u^1(y_j)\delta(t_j) - q_{t_j}(y_j)\}\, dt_j \\
&= \sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} \int_{\{\sum_{j=1}^k t_j \leq T\}} \prod_{j\in B} q_{t_j}(y_j) \prod_{j\in B^c} u^1(y_j)\delta(t_j) \prod_{j=1}^k dt_j \\
&= \sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} \left(\int_{\{\sum_{j\in B} t_j \leq T\}} \prod_{j\in B} q_{t_j}(y_j)\, dt_j\right)\left(\prod_{j\in B^c} u^1(y_j)\right) \\
&= \sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} \left(\int_{\{\sum_{j\in B} t_j \leq T\}} \prod_{j\in B} q_{t_j}(y_j)\, dt_j\right)\left(\int \prod_{j\in B^c} q_{t_j}(y_j)\, dt_j\right) \\
&= \sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} \int_{\{\sum_{j\in B} t_j \leq T\}} \prod_{j=1}^k q_{t_j}(y_j)\, dt_j \\
&= \int \left(\sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t})\right) \prod_{j=1}^k q_{t_j}(y_j)\, dt_j
\end{aligned}$$

where $T(B) = \{(t_1, \dots, t_k) \mid \sum_{j\in B} t_j \leq T\}$ and $\bar{t} = (t_1, \dots, t_k)$. Thus

$$\begin{aligned}
&|\int_{\{\sum_{j=1}^k t_j \leq T\}} \prod_{j=1}^k \{u^1(y_j)\delta(t_j) - q_{t_j}(y_j)\}\, dt_j| \\
&\qquad \leq \int | \sum_{B\subseteq\{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t})| \prod_{j=1}^k q_{t_j}(0)\, dt_j
\end{aligned} \tag{5.8}$$

and it suffices to bound this last integral over sets

$$C(A) = \{(t_1, \dots, t_k) \,|\, t_j \le T \text{ if and only if } j \in A\}$$

for all $A \subseteq \{1, \dots, k\}$.

Since $\bar{t} \in C(A)$ implies that

$$\sum_{B \subseteq \{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t}) = \sum_{B \subseteq A} (-1)^{|B|} 1_{T(B)}(\bar{t}),$$

and if $t_j \ge T$ we can integrate $\int_T^\infty q_{t_j}(0)\, dt_j = Q_T$, we see that it suffices to show that

$$\text{(5.9)} \qquad \int_{D_T} \Big| \sum_{B \subseteq \{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t}) \Big| \prod_{j=1}^{k} q_{t_j}(0)\, dt_j \le C(Q_T)^k,$$

where $D_T = C(\{1, \dots, k\}) = \cap_{j=1}^{k} \{\bar{t} \,|\, t_j \le T\}$. Furthermore, by symmetry, it suffices to show that

$$\text{(5.10)} \quad \int_{\{t_1 \le \dots \le t_k \le T\}} \Big| \sum_{B \subseteq \{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t}) \Big| \prod_{j=1}^{k} q_{t_j}(0)\, dt_j \le C(Q_T)^k.$$

Note that if $t_1 \ge T/(2n)$, (5.10) follows immediately as above using the regular variation of Q_T which implies that $Q_{T/(2n)} \le CQ_T$. Hence, we need only show

$$\text{(5.11)} \quad \int_{\substack{\{t_1 \le \dots \le t_k \le T\} \\ t_1 < T/(2n)}} \Big| \sum_{B \subseteq \{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t}) \Big| \prod_{j=1}^{k} q_{t_j}(0)\, dt_j \le C(Q_T)^k.$$

Observe that

$$\sum_{B \subseteq \{1,\dots,k\}} (-1)^{|B|} 1_{T(B)}(\bar{t}) = \sum_{A \subseteq \{2,\dots,k\}} (-1)^{|A|} \left(1_{T(A)}(\bar{t}) - 1_{T(A \cup \{1\})}(\bar{t}) \right),$$

hence we are reduced to proving (5.11) with integration restricted to sets of the form $D(A) = \{(t_1, \dots, t_k) \,|\, 1_{T(A)}(\bar{t}) - 1_{T(A \cup \{1\})}(\bar{t}) \ne 0\}$, i.e. $D(A) = \{(t_1, \dots, t_k) \,|\, \sum_A t_j \le T,\ \sum_{A \cup \{1\}} t_j > T\}$, and clearly we must have $A \ne \emptyset$. Let m denote the largest index in A. Then, if $A' = A - \{m\}$, we have

$$\text{(5.12)} \qquad T - \sum_{j \in A'} t_j - t_1 \le t_m \le T - \sum_{j \in A'} t_j.$$

Furthermore, since $\sum_{A\cup\{1\}} t_j > T$ implies that $t_m > T/n$, we have from (5.12) that

$$T/n < T - \sum_{j\in A'} t_j,$$

while $t_1 < T/(2n)$ then implies that

$$T/(2n) < T - \sum_{j\in A'} t_j - t_1. \tag{5.13}$$

Using the regular variation of Q_T once more we bound

$$\begin{aligned} &\int_{t_m \le t_{m+1} \le \cdots \le t_k} \prod_{j=m+1}^{k} q_{t_j}(0)\, dt_j \\ &\le \prod_{j=m+1}^{k} \int_{t_m}^{\infty} q_{t_j}(0)\, dt_j \le C(Q_{t_m})^{k-m} \le C(Q_T)^{k-m} \end{aligned} \tag{5.14}$$

We can then bound the dt_m integral by using (5.13) and the regular variation of $q_{t_m}(0)$ to obtain

$$\begin{aligned} &\int_{T-\sum_{j\in A'} t_j - t_1}^{T-\sum_{j\in A'} t_j} q_{t_m}(0)\, dt_m \\ &\le t_1 \left(\sup_{T-\sum_{j\in A'} t_j - t_1 \le t_m \le T - \sum_{j\in A'} t_j} q_{t_m}(0) \right) \\ &\le C t_1 q_T(0). \end{aligned} \tag{5.15}$$

On the other hand, the $dt_2 \cdots dt_{m-1}$ integral is bounded as in (5.14)

$$\int_{t_1 \le t_2 \le \cdots \le t_{m-1}} \prod_{j=2}^{m-1} q_{t_j}(0)\, dt_j \le C(Q_{t_1})^{m-2} \tag{5.16}$$

and using (5.15) and (5.16) and the regular variation of $q_{t_1}(0)$ we see that we can bound (5.11) by

$$\begin{aligned} &(Q_T)^{k-m} q_T(0) \int_{t_1 \le T} t_1 (Q_{t_1})^{m-2} q_{t_1}(0)\, dt_1 \\ &\le C(Q_T)^{k-m} q_T(0) \int_{t_1 \le T} (Q_{t_1})^{m-1}\, dt_1 \\ &\le C(Q_T)^{k-m} q_T(0) T (Q_T)^{m-1} \\ &\le C(Q_T)^k. \end{aligned} \tag{5.17}$$

The second inequality follows from the fact that $(Q_t)^{m-1}$ is regularly varying at $t = 0$ of index > -1. To see this, let β be the index of regular variation of ψ. This implies that $u^1(x)$ is regularly varying at $x = 0$ of index $\beta - d$. As we saw in the introduction, $\mathcal{G}^{2n}$ non-empty implies $u^1(x) \in L^{2n}(R^d, dx)$, so that $2n(\beta - d) \leq d$ and therefore $2n(\beta/d - 1) \leq 1$. The use of polar coordinates together with the standard Tauberian Theorem, Theorem 1.7.1 of [3], shows that $q_s(0)$ is regularly varying of index $-\beta/d$, hence Q_s is regularly varying of index $1 - \beta/d$. Since $m - 1 < 2n$, our claim is established. This completes the proof of Lemma 8.

We now return to (5.5). Assume first that $A = B_s$, so that, using (5.7) with $T_{B_s^c} = t - \sum_{B_s^c} t_p$ we have the integral

(5.18)
$$\int\int_{\{\sum_{B_s^c} t_p \leq t\}} \left(\int \left(\sum_{A \subseteq B_s} (-1)^{|A|} 1_{T_{B_s^c}(A)}(\bar{t}) \right) \prod_{p \in B_s} q_{t_p}(y_{s(p),c(p)})\, dt_p \right)$$
$$\prod_{p \in B_s^c} q_{t_p}\Big(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}\Big)$$
$$\prod_{B_s^c} dt_p \prod_{i=1}^{2} \prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\, dy_{i,j}\, d\mu(x_1)\, d\nu(x_2).$$

This differs from the integral

$$\text{(5.19)} \quad \int\int_{\{\sum_{B_s^c} t_p \leq t\}} \left(\int \left(\sum_{A \subseteq B_s} (-1)^{|A|} 1_{T_{B_s^c}(A)}(\bar{t}) \right) \prod_{p \in B_s} q_{t_p}(0)\, dt_p \right)$$
$$\prod_{p \in B_s^c} q_{t_p}\Big(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}\Big)$$
$$\prod_{B_s^c} dt_p \prod_{i=1}^{2} \prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\, dy_{i,j}\, d\mu(x_1)\, d\nu(x_2)$$

by a sum of terms similar to (5.18), in each of which the product $\prod_{p \in B_s} q_{t_p}(y_{s(p),c(p)})$ has some of the of the $y_{s(p),c(p)}$'s replaced by 0, and one factor $q_{t_p}(y_{s(p),c(p)})$ replaced by the difference $q_{t_p}(0) - q_{t_p}(y_{s(p),c(p)})$.

Using once again polar coordinates together with the standard Tauberian Theorem, Theorem 1.7.1 of [3], we have that

(5.20)
$$\begin{aligned}|q_t(0)-q_t(y)| &\le \int |1-e^{ipy}|e^{-t\psi(p)}\,dp\\ &\le |y|\int |p|e^{-t\psi(p)}\,dp\\ &\le C\frac{|y|}{\phi(t)}q_t(0)\end{aligned}$$

where $\phi(t)$ is regularly varying of index $1/\beta$ at $t=0$. (Actually, the precise index of $\phi(t)$ will not be important for us).

We have seen above that $(Q_s)^{2n-1}$ is integrable at $s=0$, so that we can find a function $h(s)\le 1$, slowly varying at $s=0$, with $h(s)\searrow 0$ as $s\searrow 0$ and such that

(5.21)
$$\int_0^T (\widetilde{Q}(s))^{2n-1}\,ds<\infty$$

with
$$\widetilde{Q}(s)=\int_s^\infty \frac{1}{h(\phi(t))}q_t(0)\,dt.$$

Interpolating between (1.13) and the trivial bound $|q_t(0)-q_t(y)|\le q_t(0)$ we have

(5.22)
$$|q_t(0)-q_t(y)|\le C\frac{h(|y|)}{h(\phi(t))}q_t(0).$$

More precisely, if $|y|\le\phi(t)$ use the bound $|y|/\phi(t)\le Ch(|y|)/h(\phi(t))$ which follows from $|y|/h(|y|)\le C\phi(t)/h(\phi(t))$ and the fact that $x/h(x)$ is regularly varying of index 1, while if $\phi(t)\le|y|$ use the bound $1\le h(|y|)/h(\phi(t))$.

Using the analysis of Lemma 8 we find that the difference between (5.18) and (5.19) can be bounded by integrals of the form

(5.23)
$$\begin{aligned}h(|\epsilon|)\int\int_{\{\sum_{B_s^c}t_p\le t\}} &(\widetilde{Q}(t-\sum_{B_s^c}t_p))^{|B_s|}\\ &\prod_{p\in B_s^c} q_{t_p}(x_{s(p)}+\sum_{j=1}^{c(p)}y_{s(p),j}-x_{s(p-1)}-\sum_{j=1}^{c(p-1)}y_{s(p-1),j})\\ &\prod_{B_s^c}dt_p\prod_{i=1}^{2}\prod_{j=1}^{n}f_{\epsilon_i}(y_{i,j})\,dy_{i,j}\,d\mu(x_1)\,d\nu(x_2)\end{aligned}$$

where $\epsilon = \max(\epsilon_1, \epsilon_2)$. If $\sum_{B_s^c} t_p < t/2$, we bound (5.23) by

(5.24)

$$Ch(|\epsilon|)(\widetilde{Q}(t/2))^{|B_s|} \int \prod_{p \in B_s^c} u^1(x_{s(p)} + \sum_{j=1}^{c(p)} y_{s(p),j} - x_{s(p-1)} - \sum_{j=1}^{c(p-1)} y_{s(p-1),j}) \prod_{i=1}^{2} \prod_{j=1}^{n} f_{\epsilon_i}(y_{i,j})\, dy_{i,j}\, d\mu(x_1)\, d\nu(x_2)$$

which converges to 0 as $\epsilon_1, \epsilon_2 \to 0$. On the other hand, if $\sum_{B_s^c} t_p \geq t/2$, then we can find $p' \in B_s^c$ with $t_{p'} \geq t/(4n)$. We bound the $q_{t_{p'}}$ factor in (5.23) by $q_{t_{p'}}(0)$, integrate over $t_{p'} \geq t/(4n)$, and in this way bound (5.23) by a similar integral in which we have one extra power of $\widetilde{Q}$ but have eliminated the q_{t_p} factor. This can be continued, and eventually shows that (5.23) converges to 0 as $\epsilon_1, \epsilon_2 \to 0$. We will briefly describe how to handle the last step in the 'worst-case scenario'.

$$\begin{aligned} (5.25)\quad & \int_0^t (\widetilde{Q}(t-s))^{2n-1} q_s(x)\, ds \\ & = \int_0^{t/2} (\widetilde{Q}(t-s))^{2n-1} q_s(x)\, ds + \int_{t/2}^{t} (\widetilde{Q}(t-s))^{2n-1} q_s(x)\, ds \\ & \leq (\widetilde{Q}(t/2))^{2n-1} u^1(x) + q_{t/2}(0) \int_0^t (\widetilde{Q}(s))^{2n-1}\, ds \\ & \leq c(u^1(x) + 1) \end{aligned}$$

where we used (5.21). This analysis also shows that (5.19) converges as $\epsilon_1, \epsilon_2 \to 0$.

We now return to the general case of (5.5) in which $B_s - A \neq \emptyset$. We rely heavily on the proof of Lemma 3.4 and 3.5 of [12]. As in that paper, if $|B_s - A| > 2$, we write out all but two of the $\mathcal{D}$ differences. If $B_s - A$ contains some p's with $s(p) = 1$ and some with $s(p) = 2$, we must be careful to retain one $\mathcal{D}$ difference for some p with $s(p) = 1$, and one for some p with $s(p) = 2$. Again expanding as in [12], we can assume that the $\mathcal{D}$ differences are now attached to one or two q_{t_p} factors with $p \in B_s^c$. We then use the basic assumptions of Class B processes (1.15), to bound our integral in terms of an integral which resembles the $A = B_s$ case of (5.5), except that on the one hand the divergent $u^1(y_{s(p),c(p)})$ factors for

$p \in B_s - A$ have now been replaced with factors that converge to 0 as $\epsilon_1, \epsilon_2 \to 0$, while on the other hand we introduce a new factor $\phi(x_1 - x_2)$ which has no effect on convergence of the $d\mu(x_1)\, d\nu(x_2)$ integral. Then using Lemma 8 as above we bound the remaining integral, showing that in fact any term in (5.5) with $B_s - A \neq \emptyset$ converges to 0 as $\epsilon_1, \epsilon_2 \to 0$. This completes the proof of Lemma 7

6. A DOOB-MEYER TYPE DECOMPOSITION

For any $n \geq 2$, let

$$\begin{aligned}(6.1)\quad & \Lambda_{n-1,\epsilon}(\mu; t) \\ &= \int\int_{\{0 \leq t_1 \leq \cdots t_{n-1} \leq t\}} f_{\epsilon,x}(X(t_1)) \prod_{j=2}^{n-1} f_\epsilon(X(t_j) - X(t_{j-1})) \\ &\qquad U^1 f_\epsilon(X(t) - X(t_{n-1}))\, dt_1 \cdots dt_{n-1}\, d\mu(x)\end{aligned}$$

and

$$\begin{aligned}(6.2)\quad & \Gamma_{n-1,\epsilon}(\mu; t) \\ &= \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} (u_\epsilon^1(0))^k \Lambda_{n-k-1,\epsilon}(\mu; t) \\ &= \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} (u_\epsilon^1(0))^k \int\int_{\{0 \leq t_1 \leq \cdots t_{n-k-1} \leq t\}} f_{\epsilon,x}(X(t_1)) \\ &\qquad \prod_{j=2}^{n-k-1} f_\epsilon(X(t_j) - X(t_{j-1})) U^1 f_\epsilon(X(t) - X(t_{n-k-1})) \\ &\qquad dt_1 \cdots dt_{n-k-1}\, d\mu(x).\end{aligned}$$

The following essentially combinatoric Lemma is the key to proving Lemma 2 and Theorem 3.

LEMMA 9

$$E^y(\gamma_{n,\epsilon}(\mu) | \mathcal{F}_t) = \gamma_{n,\epsilon}(\mu; t) + \Gamma_{n-1,\epsilon}(\mu; t) \tag{6.3}$$

Proof of Lemma 9. – It will help clarify the proof if we let the ϵ's in (1.1) vary from factor to factor. Thus we introduce

$$(6.4)\quad \alpha_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) \stackrel{def}{=} \int\int_{\{0\le t_1\le\cdots\le t_n\le t\}} f_{\epsilon_1}(X(t_1)-x)\prod_{j=2}^{n} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\, dt_1\cdots dt_n\, d\mu(x)$$

and the related approximate renormalized intersection local time

$$(6.5)\qquad \gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) = \sum_{A\subseteq\{2,\ldots,n\}} (-1)^{|A|}\prod_{j\in A} u^1_{\epsilon_j}(0)\alpha_{\epsilon_1,\epsilon_{A^c}}(\mu;t)$$

where for any set $A=\{i_1,\ldots,i_k\}\subseteq\{2,\ldots,n\}$ we let $\epsilon_A=(\epsilon_{i_1},\ldots,\epsilon_{i_k})$ and A^c will refer to the complement of A with respect to $\{2,\ldots,n\}$. We let $l(A)=i_k$ denote the largest element in A

Similarly, for $n\ge 2$, we introduce

$$(6.6)\ \Lambda_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) = \int\int_{\{0\le t_1\le\cdots t_{n-1}\le t\}} f_{\epsilon_1,x}(X(t_1))\prod_{j=2}^{n-1} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\, U^1 f_{\epsilon_n}(X(t)-X(t_{n-1}))\, dt_1\cdots dt_{n-1}\, d\mu(x)$$

and

$$(6.7)$$

$$\Gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) = \sum_{A\subseteq\{2,\ldots,n-1\}} (-1)^{|A|}\left(\prod_{j\in A} u^1_{\epsilon_j}(0)\right)\Lambda_{\epsilon_1,\epsilon_{A^c},\epsilon_n}(\mu;t)$$

$$= \sum_{A\subseteq\{2,\ldots,n-1\}} (-1)^{|A|}\left(\prod_{j\in A} u^1_{\epsilon_j}(0)\right)\int\int_{\{0\le t_1\le\cdots t_{l(A^c)}\le t\}} f_{\epsilon_1,x}(X(t_1)) \prod_{j\in A^c} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))U^1 f_{\epsilon_n}(X(t)-X(t_{l(A^c)}))\, dt_1\cdots dt_{l(A^c)}\, d\mu(x)$$

where, naturally, A^c is taken with respect to $\{2,\dots,n-1\}$. We also set

$$\Lambda_{\epsilon_1}(\mu;t)=\int U^1 f_{\epsilon_1}(X(t)-x)\,d\mu(x). \tag{6.8}$$

In the following, $\widetilde{X}$ denotes an independent copy of X, and $\widetilde{E}^w$ denotes expectation with respect to $\widetilde{X}$ starting at w. From the definition (6.4) of $\alpha_{\epsilon_1,\dots,\epsilon_k}(\mu)$ we see that

(6.9)

$$\begin{aligned}
&E^y(\alpha_{\epsilon_1,\dots,\epsilon_n}(\mu)|\,\mathcal{F}_t)\\
&=\sum_{i=0}^{k} E^y\Big(\int\int_{\{0\le t_1\le\cdots t_i\le t\le t_{i+1}\le\cdots\le t_k\}}\\
&\qquad f_{\epsilon_1,x}(X(t_1))\prod_{j=2}^{k} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\,dt_1\cdots dt_k\,d\mu(x)|\,\mathcal{F}_t\Big)\\
&=\sum_{i=0}^{k}\int\int_{\{0\le t_1\le\cdots t_i\le t\le t_{i+1}\le\cdots\le t_k\}} f_{\epsilon_1,x}(X(t_1))\prod_{j=2}^{i} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\\
&\qquad E^y\left\{\prod_{j=i+1}^{k} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\,\middle|\,\mathcal{F}_t\right\}dt_1\cdots dt_k\,d\mu(x)\\
&=\sum_{i=0}^{k}\int\int_{\{0\le t_1\le\cdots t_i\le t\le t_{i+1}\le\cdots\le t_k\}} f_{\epsilon_1,x}(X(t_1))\prod_{j=2}^{i} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\\
&\qquad \widetilde{E}^{X(t)}\left\{f_{\epsilon_{i+1}}(\widetilde{X}(t_{i+1}-t)-X(t_i))\prod_{j=i+2}^{k} f_{\epsilon_j}(\widetilde{X}(t_j-t)-\widetilde{X}(t_{j-1}-t))\right\}\\
&\qquad\qquad dt_1\cdots dt_k\,d\mu(x)\\
&=\sum_{i=0}^{k}\int\int_{\{0\le t_1\le\cdots t_i\le t\}} f_{\epsilon_1,x}(X(t_1))\prod_{j=2}^{i} f_{\epsilon_j}(X(t_j)-X(t_{j-1}))\\
&\qquad U^1 f_{\epsilon_{i+1}}(X(t)-X(t_i))\left(\prod_{j=i+2}^{k} u^1_{\epsilon_j}(0)\right)dt_1\cdots dt_i\,d\mu(x)\\
&=\alpha_{\epsilon_1,\dots,\epsilon_k}(\mu;t)+\sum_{i=0}^{k-1}\Lambda_{\epsilon_1,\dots,\epsilon_{i+1}}(\mu;t)\prod_{j=i+2}^{k} u^1_{\epsilon_j}(0).
\end{aligned}$$

Consequently, adopting the notation that for any set $B=\{j_1,\dots,j_k\}\subseteq\{2,\dots,n\}$ we let $B_i=(j_1,\dots,j_{i-1},j_i)$, and $B^i=(j_i,j_{i+1},\dots,j_k)$, we have

$$
\begin{aligned}
(6.10)\quad & E^y(\gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu)|\mathcal{F}_t) \\
&= \sum_{A\subseteq\{2,\ldots,n\}} (-1)^{|A|} \prod_{j\in A} u^1_{\epsilon_j}(0) E^y(\alpha_{\epsilon_1,\epsilon_{A^c}}(\mu)|\mathcal{F}_t) \\
&= \sum_{A\subseteq\{2,\ldots,n\}} (-1)^{|A|} \prod_{j\in A} u^1_{\epsilon_j}(0) \\
&\qquad \times \left\{ \alpha_{\epsilon_1,\epsilon_{A^c}}(\mu;t) + \sum_{i=0}^{|A^c|} \Lambda_{\epsilon_1,\epsilon_{A^c_i}}(\mu;t) \prod_{j\in A^{c,i+1}} u^1_{\epsilon_j}(0) \right\} \\
&= \gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) + \sum_{A\subseteq\{2,\ldots,n\}} (-1)^{|A|} \\
&\qquad \times \prod_{j\in A} u^1_{\epsilon_j}(0) \sum_{i=0}^{|A^c|} \Lambda_{\epsilon_1,\epsilon_{A^c_i}}(\mu;t) \prod_{j\in A^{c,i+1}} u^1_{\epsilon_j}(0) \\
&= \gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu;t) \\
&\quad + \sum_{B\subseteq\{2,\ldots,n\}} \left\{ (-1)^{|B^c\cap\{2,\ldots,l(B)-1\}|} \prod_{j\in B^c\cap\{2,\ldots,l(B)-1\}} u^1_{\epsilon_j}(0) \right. \\
&\qquad \left. \sum_{A_2\subseteq\{l(B)+1,\ldots,n\}} (-1)^{|A_2|} \prod_{j=l(B)+1}^{n} u^1_{\epsilon_j}(0) \right\} \Lambda_{\epsilon_1,\epsilon_B}(\mu;t).
\end{aligned}
$$

Since $\sum_{A\subseteq D}(-1)^{|A|} = 0$ for any nonempty set D, we see that in the last display B will not contribute unless we have $l(B) = n$. We can then rewrite the sum in (6.10) as

$$
\begin{aligned}
(6.11)\quad & \sum_{\substack{B\subseteq\{2,\ldots,n\}\\ l(B)=n}} (-1)^{|B^c\cap\{2,\ldots,n-1\}|} \left(\prod_{j\in B^c\cap\{2,\ldots,n-1\}} u^1_{\epsilon_j}(0) \right) \Lambda_{\epsilon_1,\epsilon_B}(\mu;t) \\
&= \sum_{A\subseteq\{2,\ldots,n-1\}} (-1)^{|A|} \left(\prod_{j\in A} u^1_{\epsilon_j}(0) \right) \Lambda_{\epsilon_1,\epsilon_{A^c},\epsilon_n}(\mu;t) \\
&= \Gamma_{\epsilon_1,\ldots,\epsilon_n}(\mu;t)
\end{aligned}
$$

This completes the proof of Lemma 9.

Proof of Lemma 2. – Let

(6.12)

$$\Gamma_{n-1,\epsilon}(v;\mu^\delta;t) = \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(u^1_\epsilon(0))^k \int\int_{\{0\le t_1\le\cdots t_{n-k-1}\le t\}} f_{\epsilon,x}(X(t_1)) \prod_{j=2}^{n-k-1} f_\epsilon(X(t_j)-X(t_{j-1}))u^1(v-X(t_{n-k-1})))\,dt_1\cdots dt_{n-k-1}\,d\mu^\delta(x)$$

An easy modification of [14] shows that $\Gamma_{n-1,\epsilon}(v;\mu^\delta;t)$ converges a.s. locally uniformly in $v\in R^d$ to $\gamma_{n-1}(u^1_v\cdot\mu^\delta;t)$ as $\epsilon\to 0$. By Lemma 9 this implies Lemma 2 for $k=n$. The general case follows by replacing n in Lemma 9 by k and μ by $\prod_{j=1}^{n-k} u^1_{v_j}\cdot\mu^\delta_x$.

Proof of Theorem 3. – This now follows easily from Lemma 2 on letting $\delta\to 0$ and using the proof of Theorem 1.

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