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Self-collisions of superprocesses: renormalization and limit theorems

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Abstract

In this paper, we study a renormalized self-collision local time for superprocesses over stable processes and classical diffusions. When the renormalization breaks down, we obtain limit theorems. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

With any nice Markov process z_t in $\mathbb{R}^d$ we associate a new Markov Z_t taking values in the space of finite measures on $\mathbb{R}^d$. The process Z_t is called the superprocess over z_t, and we refer to Dynkin (1988a) for an introduction to superprocesses and for further references.

We will use the notation

$$\langle \varphi, Z_t \rangle = \int \varphi(x) Z_t(\mathrm{d}x),$$

$$\langle f(x,y),\ Z_s(\mathrm{d}x)Z_t(\mathrm{d}y)\rangle = \iint f(x,y) Z_s(\mathrm{d}x)Z_t(\mathrm{d}y).$$

Throughout this paper we assume that the initial measure $Z_0 = \mu$ has a bounded and integrable density with respect to Lebesgue measure. Also we use $|v|$ for the mass of a measure v. Our starting point is the formal expression

$$\int_0^T \langle \delta(x-y), Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\, \mathrm{d}s \tag{1.1}$$

which intuitively should measure the ‘self-collisions’ of Z_s. In Eq. (1.1), δ is the Dirac delta ‘function’. In an attempt to make Eq. (1.1) rigorous, we replace δ by

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an approximate delta function. Let $f(x)\geqslant 0$ be a continuous symmetric function with support in the unit ball, and such that $\int f(x)\,\mathrm{d}x=1$. Set

$$f_\varepsilon(x)=\frac{1}{\varepsilon^d}f\left(\frac{x}{\varepsilon}\right)$$

and replace Eq. (1.1) by

$$\int_0^T \langle f_\varepsilon(x-y), Z_s(\mathrm{d}x)\ Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s. \tag{1.2}$$

We will describe the behavior of Eq. (1.2) as $\varepsilon\to 0$.

To be specific, let us first take Z_t to be the superprocess over Brownian motion in $\mathbb{R}^d$. If $d=1$ there are no problems with the $\varepsilon\to 0$ limit. However, if $d\geqslant 2$, the case we study here, Eq. (1.2) will typically blow up as $\varepsilon\to 0$. In Theorem 1 we will show that if $d=2$ or 3, then Eq. (1.2) can be renormalized, i.e. by subtracting a singular term which does not involve collisions, we can obtain a nontrivial limit.

This is the analogue of a result for self-intersections of Brownian motion in $\mathbb{R}^2$ which goes back to Varadhan (1969); see also Le Gall (1985), Rosen (1986), Yor (1985a) and Dynkin (1988b). See also our work on the self-intersections of superprocesses (Rosen, 1992), which initiated many of the techniques used in this paper. The renormalized intersection local time for Brownian motion in $\mathbb{R}^2$ turns out to be the right tool for analyzing fluctuations of the Wiener Sausage (see Le Gall, 1986b; Chavel et al., 1991; Wienryb, 1987) and the range of random walks (Le Gall, 1986a; Le Gall-Rosen, 1991). It is our hope that the renormalized collision local time of Theorem 1 will find similar applications to the study of measure-valued processes. In this regard see the recent paper of Evans and Perkins (1997).

When $d=4$, we can no longer obtain a renormalized collision local time. However, Theorem 1 shows that a suitably scaled version converges in distribution. This is the analogue of Yor's theorem for Brownian motion in $\mathbb{R}^3$, (Yor, 1985b; Rosen, 1988).

We use B_t to denote a real Brownian motion independent of our superprocess.

Theorem 1. *Let Z_t denote the superprocess over Brownian motion in $\mathbb{R}^d$, and set*

$$\gamma_\varepsilon(T)=\int_0^T \langle f_\varepsilon(x-y), Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s-2\int_0^T \varphi_\varepsilon(T-s)|Z_s|\,\mathrm{d}s, \tag{1.3}$$

where

$$\varphi_\varepsilon(t)=\int_0^t\left(\iint p_r(x)f_\varepsilon(x-y)p_r(y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}r \tag{1.4}$$

and

$$p_s(y)=\frac{\mathrm{e}^{-y^2/2s}}{(2\pi s)^{d/2}}$$

is the transition density for Brownian motion in $\mathbb{R}^d$.

If $d=2$ or 3, then $\gamma_\varepsilon(T)$ converges in L^2 as $\varepsilon\to 0$.

If $d=4$, *then* $\gamma_\varepsilon(T)/\lg(1/\varepsilon)$ *converges weakly to* B_{M_T} *where*

$$M_T=\frac{1}{\pi^4}\int_0^T |Z_s|\,\mathrm{d}s \tag{1.5}$$

Remarks on Theorem 1.

- More generally, if $h\in C_0^\infty(\mathbb{R}^d)$ and we set

$$\gamma_\varepsilon(T,h)=\int_0^T \langle h(x)f_\varepsilon(x-y),Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle-2\int_0^T\langle\varphi_{\varepsilon,T-s},Z_s\rangle\,\mathrm{d}s,$$

where now

$$\varphi_{\varepsilon,t}(z)=\int_0^t\left(\iint h(z+x)p_r(x)f_\varepsilon(x-y)p_r(y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}r,$$

then $\gamma_\varepsilon(T,h)$ converges in L^2 for $d=2,3$ while if $d=4$

$$\frac{\gamma_\varepsilon(T,h)}{\lg(1/\varepsilon)}$$

converges weakly to $B_{M_T(h)}$ where

$$M_T(h)=\frac{1}{\pi^4}\int_0^T\langle h^2,Z_s\rangle\,\mathrm{d}s.$$

- Adler and Lewin (1991) have developed a Tanaka-like formula for the renormalized intersection local time of super-Brownian motion. Formally applying Lemma 1.3 of Adler and Lewin (1991), (Ito's formula), as in Lemma 1.4 of Adler and Lewin (1991) we obtain

$$\begin{aligned}&\langle G_\varepsilon(x-y),Z_T(\mathrm{d}x)Z_T(\mathrm{d}y)\rangle\\&\quad=\langle G_\varepsilon(x-y),Z_0(\mathrm{d}x)Z_0(\mathrm{d}y)\rangle+\int_0^T\langle\Delta G_\varepsilon(x-y),Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s\\&\qquad+2\int_0^T\langle G_\varepsilon(0),Z_s(\mathrm{d}x)\rangle\,\mathrm{d}s+2\int_0^T\langle G_\varepsilon(x-y),Z_s(\mathrm{d}x)M(\mathrm{d}s,\mathrm{d}y)\rangle,\end{aligned} \tag{1.6}$$

where $G(x)$ (see Eqs. (2.2) and (2.3)), is the 1-potential for Brownian motion in $\mathbb{R}^d$, $G_\varepsilon(x)=f_\varepsilon * G(x)$ and $M(\mathrm{d}s,\mathrm{d}y)$ is the martingale measure associated with super-Brownian motion. Setting

$$\hat\gamma(T)=\lim_{\varepsilon\to0}\int_0^T\langle f_\varepsilon(x-y),Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s-G_\varepsilon(0)\int_0^T|Z_s|\,\mathrm{d}s, \tag{1.7}$$

which is very similar to $\gamma(T)$ (see e.g. Eq. (2.8)), and using the fact that $(-\Delta/2+1)G_\varepsilon=f_\varepsilon$, Eq. (1.6) suggests that if $d=2$ or 3 we will get the Tanaka-like formula

$$\begin{aligned}2\hat\gamma(T)&=\langle G(x-y),Z_0(\mathrm{d}x)Z_0(\mathrm{d}y)\rangle-\langle G(x-y),Z_T(\mathrm{d}x)Z_T(\mathrm{d}y)\rangle\\&\quad+2\int_0^T\langle G(x-y),Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s\\&\quad+2\int_0^T\langle G(x-y),Z_s(\mathrm{d}x)M(\mathrm{d}s,\mathrm{d}y)\rangle.\end{aligned} \tag{1.8}$$

It would be interesting to justify such a formula. (We caution the reader that the super-Brownian motion in Adler and Lewin (1991) and Adler (1993) is somewhat different from the super-Brownian motion considered here which follows Dynkin (1988a).)

- Adler (1993) gives a particle picture interpretation for the renormalized intersection local time of super-Brownian motion. Using his notation we have for $\alpha \neq \beta$, $|\alpha| = |\beta|$ and $t \sim \alpha$,

$$G_\varepsilon(X_t^\alpha - X_t^\beta) = G_\varepsilon(X_{|\alpha|/|\mu|}^\alpha - X_{|\alpha|/|\mu|}^\beta) + \int_{|\alpha|/|\mu|}^t \nabla G_\varepsilon(X_s^\alpha - X_s^\beta)\,\mathrm{d}X_s^\alpha - \int_{|\alpha|/|\mu|}^t \nabla G_\varepsilon(X_s^\alpha - X_s^\beta)\,\mathrm{d}X_s^\beta + \int_{|\alpha|/|\mu|}^t \Delta G_\varepsilon(X_s^\alpha - X_s^\beta)\,\mathrm{d}s. \tag{1.9}$$

Set

$$\hat{\gamma}_\varepsilon^\mu(T) = \frac{1}{\mu^2}\sum_{\alpha,\beta}\int_0^T f_\varepsilon(X_s^\alpha - X_s^\beta)\,\mathrm{d}s - \frac{1}{\mu}\sum_\alpha \int_0^T 1_{\mathbb{R}^d}(X_s^\alpha)\,\mathrm{d}s. \tag{1.10}$$

Arguing as in Adler (1993), Eq. (1.9) leads to

$$\begin{aligned}2\hat{\gamma}_\varepsilon^\mu(T) &= \langle G_\varepsilon(x-y), Z_0^\mu(\mathrm{d}x)Z_0^\mu(\mathrm{d}y)\rangle - \langle G_\varepsilon(x-y), Z_T^\mu(\mathrm{d}x)Z_T^\mu(\mathrm{d}y)\rangle \\ &\quad + 2\int_0^T \langle G_\varepsilon(x-y), Z_s^\mu(\mathrm{d}x)Z_s^\mu(\mathrm{d}y)\rangle\,\mathrm{d}s \\ &\quad + 2\int_0^T \langle G_\varepsilon(x-y), Z_s^\mu(\mathrm{d}x)M^\mu(\mathrm{d}s,\mathrm{d}y)\rangle,\end{aligned} \tag{1.11}$$

where M^μ is the martingale measure analogous to that which is denoted Z^μ in Adler and Lewin (1991) and we used $(-\Delta + 2)G_\varepsilon = 2f_\varepsilon$ to handle the case of $\alpha = \beta$. Comparing Eqs. (1.8) and (1.11) suggests that $\hat{\gamma}_\varepsilon^\mu(T) \to \hat{\gamma}(T)$, where, as in Adler and Lewin (1991), we take $\mu \to \infty$ and $\varepsilon = \mu^{-c}$ for appropriate $c > 0$. If indeed this could be proven, it would indicate that the renormalization term is needed only to control the spurious collisions which arise from including $\beta = \alpha$ in Eq. (1.10).

Theorem 1 will be derived with the aid of the following very explicit theorem.

Theorem 2. *Let x_t be Brownian motion in $\mathbb{R}^d$ killed at an independent exponential time, and let X_t be the superprocess over x_t.*

(a) *If $d = 2$, then*

$$\int_0^\infty \langle f_\varepsilon(x-y), X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - \frac{1}{\pi}\lg\left(\frac{1}{\varepsilon}\right)\int_0^\infty |X_s|\,\mathrm{d}s \tag{1.12}$$

converges in L^2 as $\varepsilon \to 0$.

(b) *If $d = 3$, then*

$$\int_0^\infty \langle f_\varepsilon(x-y), X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - \frac{1}{2\pi}\frac{c(f)}{\varepsilon}\int_0^\infty |X_s|\,\mathrm{d}s \tag{1.13}$$

where $c(f) = \int f(x)(1/|x|)\,\mathrm{d}x$, converges in L^2 as $\varepsilon \to 0$.

(c) *If* $d=4$, *and*

$$\gamma_\varepsilon=\int_0^\infty \langle f_\varepsilon(x-y),X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s-a(\varepsilon)\int_0^\infty |X_s|\,\mathrm{d}s, \tag{1.14}$$

where

$$a(\varepsilon)=\frac{1}{2\pi^2}\left(\frac{1}{\varepsilon^2}\int f(y)\frac{1}{y^2}\mathrm{d}y-\lg\left(\frac{1}{\varepsilon}\right)\right), \tag{1.15}$$

then $\gamma_\varepsilon/\lg(1/\varepsilon)$ *converges in distribution and we have*

$$E_\mu(\mathrm{e}^{-\lambda(\gamma_\varepsilon/\lg(1/\varepsilon))})\to \mathrm{e}^{|\mu|(1/2)(1-\sqrt{1-2\lambda^2/\pi^4})} \tag{1.16}$$

for λ *small, as* $\varepsilon\to 0$.

Remark. X_t is not the same as Z_t killed at an independent exponential time!

Theorem 1 can be generalized to nice diffusions in $\mathbb{R}^d$. Let z_t be a diffusion with generator

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}. \tag{1.17}$$

If a_{ij},b_i are smooth and uniformly bounded together with their derivatives, and

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j\geqslant\delta\sum_{i=1}^{d}\lambda_i^2 \tag{1.18}$$

for some $\delta>0$, uniformly in x and λ_i, we will say that z_t is a smooth uniformly elliptic diffusion.

Theorem 3. *Let* Z_t *denote the superprocess over* z_t, *a smooth uniformly elliptic diffusion in* $\mathbb{R}^d$ *and set*

$$\gamma_\varepsilon(T)=\int_0^T \langle f_\varepsilon(x-y),Z_s(\mathrm{d}x)Z_s(\mathrm{d}y)\rangle\,\mathrm{d}s-2\int_0^T \langle\varphi_{\varepsilon,T-s},Z_s\rangle\,\mathrm{d}s, \tag{1.19}$$

where

$$\varphi_{\varepsilon,t}(z)=\int_0^t\left(\iint p_r(z,x)p_r(z,y)f_\varepsilon(x-y)\,\mathrm{d}x\mathrm{d}y\right)\mathrm{d}r \tag{1.20}$$

and $p_s(x,y)$ *is the transition density for* z_t.

If $d=2$ *or* 3, *then* $\gamma_\varepsilon(T)$ *converges in* L^2 *as* $\varepsilon\to 0$.

If $d=4$ *then* $\gamma_\varepsilon(T)/\lg(1/\varepsilon)$ *converges weakly to* B_{M_T} *where*

$$M_T=\frac{1}{\pi^4}\int_0^T\langle\psi,Z_s\rangle\,\mathrm{d}s \tag{1.21}$$

and

$$\psi(x)=\frac{1}{\det a_{ij}(x)}. \tag{1.22}$$

We now generalize Theorem 1 to symmetric stable processes of order β in $\mathbb{R}^d$. As before, only the case $\beta \leqslant d$ is of interest, since if $\beta > d$, the $\varepsilon \to 0$ limit exists.

Theorem 4. *Let Y_t denote the superprocess over the symmetric stable process y_t of order β in $\mathbb{R}^d$, and*

$$\gamma_\varepsilon(T) = \int_0^T \langle f_\varepsilon(x-y), Y_s(\mathrm{d}x) Y_s(\mathrm{d}y) \rangle \,\mathrm{d}s - 2\int_0^T \varphi_\varepsilon(T-s)|Y_s|\,\mathrm{d}s, \tag{1.23}$$

where

$$\varphi_\varepsilon(t) = \int_0^t \left(\iint p_r(x) f_\varepsilon(x-y) p_r(y)\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}r \tag{1.24}$$

and $p_s(y)$ denotes the transition density for y_t.

If $d/2 < \beta \leqslant d$, then $\gamma_\varepsilon(T)$ converges in L^2 as $\varepsilon \to 0$.

If $\beta = d/2$, then $\gamma_\varepsilon(T)/\lg(1/\varepsilon)$ converges weakly to B_{M_T} where

$$M_T = a(d) \int_0^T |Y_s|\,\mathrm{d}s \tag{1.25}$$

and

$$a(d) = \frac{2^{4-2d}}{\pi^d} \frac{1}{\Gamma^2(d/2)}$$

Sections 2–6 are devoted to Theorem 2, i.e. the superprocess over killed Brownian motion. In Section 7, we derive Theorem 1 with the aid of Theorem 2. The necessary modifications for the proofs of Theorems 3 and 4 are explained in Sections 8 and 9.

The present paper is a sequel to Rosen (1992) which studied renormalization and limit theorems for self-intersections of superprocesses. (Collisions occur at the same time, while self-intersections of the path can occur at different times.) We have tried to adhere to the structure of that paper to allow the reader easy reference. Some arguments needed for the present paper are almost identical to those in Rosen (1992), and in such cases we have simply referred the reader to that paper.

2. Theorem 2: preliminaries

Our proofs involve the calculation of moments, and in this section we derive a formula for moments of the approximate renormalized collision local time. Our starting point is Dynkin's formula (1988a)

$$E_\mu\left(\prod_{i=1}^n \langle f_i, X_{t_i} \rangle \right) = \sum_{D_n} \int \prod_{v \in V_-} \mu(\mathrm{d}y_v) \prod_{a \in A} p_{s_{f(a)} - s_{i(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v \in V_0} \mathrm{d}s_v \mathrm{d}y_v \prod_{i=1}^n f_i(z_i)\,\mathrm{d}z_i. \tag{2.1}$$

In Eq. (2.1),

$$p_s(x)=\mathrm{e}^{-s}\frac{\mathrm{e}^{-x^2/2s}}{(2\pi s)^{d/2}} \tag{2.2}$$

is the transition density for exponentially killed Brownian motion in $\mathbb{R}^d$, where by convention $p_s(x)=0$ if $s<0$. D_n is the set of directed binary graphs with n exits marked $1,2,\ldots,n$. Given such a graph, A is the set of arrows, and if the arrow $a\in A$ goes from the vertex v to w, we write $v=i(a)$, $w=f(a)$. To each vertex v we associate two variables

$$(s_v, y_v)\in\mathbb{R}_+\times\mathbb{R}^d$$

which we refer to as the time and space coordinates of v. V_- denotes the set of entrances for our graph, and if $v\in V_-$, we set $s_v\equiv 0$. If v is the exit labelled by j, $i\leqslant j\leqslant n$, we set

$$(s_v, y_v)\doteq(t_j, z_j).$$

Finally, V_0 denotes the set of internal vertices, i.e. those vertices which are neither entrances nor exits.

Let

$$G(x)=\int_0^\infty p_s(x)\,\mathrm{d}s \tag{2.3}$$

denote the Green's function for exponentially killed Brownian motion in $\mathbb{R}^d$. From Eq. (2.1) we see that

$$\begin{aligned}&E_\mu\left(\prod_{i=1}^n\int_0^\infty\langle f_i, X_{t_i}\rangle\,\mathrm{d}t_i\right)\\&\quad=\sum_{D_n}\int\prod_{v\in V_-}\mu(\mathrm{d}y_v)\prod_{a\in A}G(y_{f(a)}-y_{i(a)})\prod_{v\in V_0}\mathrm{d}y_v\prod_{i=1}^n f_i(z_i)\,\mathrm{d}z_i.\end{aligned} \tag{2.4}$$

From Eq. (2.1) it follows that

$$\begin{aligned}&E_\mu\left(\left[\int_0^\infty\langle f_\varepsilon(x-y), X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s\right]^n\right)\\&\quad=\sum_{D_{2n}}\int\prod_{v\in V_-}\mu(\mathrm{d}y_v)\prod_{a\in A}p_{s_{f(a)}-s_{i(a)}}(y_{f(a)}-y_{i(a)})\prod_{v\in V_0}\mathrm{d}s_v\mathrm{d}y_v\\&\qquad\times\prod_{i=1}^n f_\varepsilon(z_{2i}-z_{2i-1})\,\mathrm{d}z_{2i}\,\mathrm{d}z_{2v}\,\mathrm{d}r_i,\end{aligned} \tag{2.5}$$

where now the times t_{2i-1}, t_{2i} associated with the exits labeled $2i-1, 2i$ are both replaced by r_i, and we integrate r_i over $[0,\infty]$.

We will say that the pair of exits v, w are coupled if for some k we have

$$z_{2k}=y_v,\quad z_{2k-1}=y_w$$

or

$$z_{2k}=y_w, \quad z_{2k-1}=y_v. \tag{2.6}$$

We will say that a pair of exits v,w are a twin if they have the same immediate predecessor, i.e., if we can find $a,b\in A$ and a vertex u such that

$$i(a)=i(b)=u \tag{2.7}$$

and

$$f(a)=v, \qquad f(b)=w.$$

If a twin v,w are coupled, and e.g. $z_{2k}=y_v,\ z_{2k-1}=y_w$ and Eq. (2.7) holds, then we get a factor in Eq. (2.5) of the form

$$\begin{aligned}
&\int_s^\infty \iint p_{t-s}(y_v-y_u)\, p_{t-s}(y_w-y_u)\, f_\varepsilon(y_v-y_w)\,\mathrm{d}y_v\,\mathrm{d}y_w\,\mathrm{d}t \\
&\quad=\int_s^\infty \iint p_{t-s}(y_v)\, p_{t-s}(y_w)\, f_\varepsilon(y_v-y_w)\,\mathrm{d}y_v\,\mathrm{d}y_w\,\mathrm{d}t \\
&\quad=\int_0^\infty \int p_{2t}(y)\, f_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad=\frac{1}{2}\int f_\varepsilon(y)G(y)\,\mathrm{d}y.
\end{aligned} \tag{2.8}$$

Set

$$c(\varepsilon)=\int f_\varepsilon(y)G(y)\,\mathrm{d}y. \tag{2.9}$$

Then it is easy to check that

$$\begin{aligned}
&E_\mu\left(\left[\int_0^\infty \langle f_\varepsilon(x-y), X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s\right]^n\right) \\
&\quad=\sum_{C_{2n}}\int \prod_{v\in V_-}\mu(\mathrm{d}y_v)\prod_{a\in A} p_{s_{f(a)}-s_{i(a)}}(y_{f(a)}-y_{i(a)})\prod_{v\in V_0}\mathrm{d}s_v\mathrm{d}y_v \\
&\qquad\times\prod_{i=1}^n f_\varepsilon(z_{2i}-z_{2i-1})\,\mathrm{d}z_{2i}\,\mathrm{d}z_{2v}\,\mathrm{d}r_i,
\end{aligned} \tag{2.10}$$

where C_{2n} is the set of binary graphs with $2n$ labeled exits; $1,2,\ldots,2n$, such that no twin exits are coupled, i.e. no twin exits are labeled $2i-1,2i$ for any i.

Thus, the effect of the subtraction term in Eq. (2.10) is to eliminate all coupled twins. The factor 2 comes from the two possibilities in Eq. (2.6).

We now calculate the asymptotics of $c(\varepsilon)$. We first note that

$$\begin{aligned}
G*G(y)&=\int \mathrm{d}x\int_0^\infty\int_0^\infty p_s(y-x)p_t(x)\,\mathrm{d}s\,\mathrm{d}t \\
&=\int_0^\infty\int_0^\infty p_{s+t}(y)\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^t p_t(y)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty t p_t(y)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-t}\frac{\mathrm{e}^{-|y|^2/2r}}{(2\pi t)^{(d-2)/2}} \\
&\doteq \frac{1}{2\pi} g(y)
\end{aligned} \tag{2.11}
$$

where $g(y), y\in\mathbb{R}^d$, with obvious notation, corresponds to the Green's function for killed Brownian motion in $d-2$ dimensions.

If $d=2$, it is known that for $|y|\leqslant\frac{1}{2}$,

$$G(y)=\frac{1}{\pi}\left[\lg\left(\frac{1}{|y|}\right)+\lg(\sqrt{2})-\kappa\right]+\mathrm{O}(|y|), \tag{2.12}$$

where κ is Euler's constant. Hence

$$
\begin{aligned}
c(\varepsilon) &= \int f_\varepsilon(y)G(y)\,\mathrm{d}y \\
&= \frac{1}{\pi}\int f_\varepsilon(y)\left(\lg\left(\frac{1}{|y|}\right)\mathrm{d}y+\lg(\sqrt{2})-\kappa\right)+\mathrm{O}(|y|))\,\mathrm{d}y \\
&= \frac{1}{\pi}\left(\lg\left(\frac{1}{\varepsilon}\right)+\int f(y)\lg\left(\frac{1}{|y|}\right)\mathrm{d}y+\lg(\sqrt{2})-\kappa\right)+\mathrm{O}(\varepsilon).
\end{aligned} \tag{2.13}
$$

If $d=3$, it is known that

$$G(y)=\frac{1}{2\pi}\frac{\mathrm{e}^{-|y|}}{|y|}. \tag{2.14}$$

Hence

$$
\begin{aligned}
c(\varepsilon) &= \frac{1}{2\pi}\int f_\varepsilon(y)\frac{\mathrm{e}^{-|y|}}{|y|}\,\mathrm{d}y \\
&= \frac{1}{2\pi\varepsilon}\int \frac{\mathrm{e}^{-\varepsilon|y|}}{|y|}f(y)\,\mathrm{d}y \\
&= \frac{1}{2\pi\varepsilon}\int f(y)\frac{\mathrm{d}y}{|y|}-\frac{1}{2\pi}+\mathrm{O}(\varepsilon).
\end{aligned} \tag{2.15}
$$

Finally, for $d=4$, let us analyze $G(x)$ using $G_0(x)=(1/2\pi^2)1/|x|^2$ the zero-potential for Brownian motion in $\mathbb{R}^4$. Iterating the resolvent equation we find

$$
\begin{aligned}
G_0(x)-G(x) &= G * G_0(x) \\
&= G * G(x)+G * G * G_0(x).
\end{aligned} \tag{2.16}
$$

By Eqs. (2.11) and (2.12), we know that

$$G * G(x) = \frac{1}{2\pi^2} \lg\left(\frac{1}{|x|}\right) + \mathrm{O}(1), \quad |x| < 1 \tag{2.17}$$

and it is easy to see that

$$G * G * G_0(x) = \mathrm{O}(1)$$

so that for $|x| < \frac{1}{2}$

$$\begin{aligned} G(x) &= G_0(x) - \frac{1}{2\pi^2} \lg\left(\frac{1}{|x|}\right) + 0(1) \\ &= \frac{1}{2\pi^2}\frac{1}{x^2} - \frac{1}{2\pi^2} \lg\left(\frac{1}{|x|}\right) + 0(1). \end{aligned} \tag{2.18}$$

Hence

$$\begin{aligned} c(\varepsilon) &= \int f_\varepsilon(y) G(y)\,\mathrm{d}y \\ &= \frac{1}{2\pi^2}\frac{1}{\varepsilon^2}\int \frac{f(y)}{y^2}\,\mathrm{d}y - \frac{1}{2\pi^2}\lg\left(\frac{1}{\varepsilon}\right) + 0(1), \quad d = 4. \end{aligned} \tag{2.19}$$

We also note for future reference that, as in Eq. (2.1),

$$\begin{aligned} G * G * G(y) &= \int_0^\infty \cdot \int_0^\infty \int_0^\infty p_{r+s+t}(y)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty \int_0^t \int_0^s p_t(y)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty \frac{t^2}{2} p_t(y)\,\mathrm{d}t. \end{aligned} \tag{2.20}$$

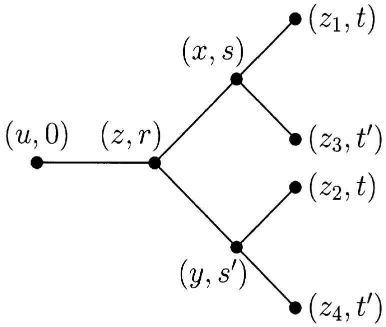
3. Theorem 2: the second moment

In this section, we compute the asymptotics of

$$I(\varepsilon) = E_\mu\left[\left(\int_0^\infty \langle f_\varepsilon(x-y),\ X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s\right)^2\right]. \tag{3.1}$$

By Eq. (2.10) we obtain a contribution from each binary graph with four exits, such that no twin exits are coupled.

We first sketch the possible graphs and write down their contribution. Later we will work out the combinatoric factors. We sometimes use the abbreviation $\mathrm{d}x \ldots$ to indicate integration over all variables.



Graph 1

$$\int \mu(\mathrm{d}u)\, p_r(z-u)\, p_{s-r}(x-z)\, p_{s'-r}(y-z)\, p_{t-s}(x-z_1)\, p_{t'-s}(x-z_3)\, p_{t-s'}(y-z_2)$$

$$p_{t'-s'}(y-z_4) f_\varepsilon(z_1-z_2) f_\varepsilon(z_3-z_4)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\bar{z}\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}s'\, \mathrm{d}t\, \mathrm{d}t'$$

$$= \int p_r(z-u)\, \mathrm{d}z\, \mathrm{d}\mu(u) \int p_{s-r}(x)\, p_{s'-r}(y)\, p_{t-s'+t-s} * f_\varepsilon(x-y)\, p_{t'-s'+t'-s}$$

$$* f_\varepsilon(x-y)\, \mathrm{d}x \ldots$$

$$= \int p_r(z)\, \mathrm{d}z\, \mathrm{d}\mu(u) \int p_{s-r}(x)\, p_{s'-r'}(y)\, p_{t-s'+t-s} * f_\varepsilon(x-y)\, p_{t'-s'+t'-s}$$

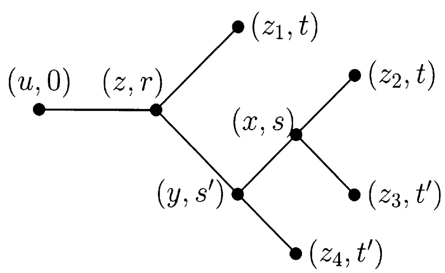
$$* f_\varepsilon(x-y)\, \mathrm{d}x \ldots$$

$$= |\mu| \int p_{s-r}(x+y)\, p_{s'-r}(y)\, p_{t-s'+t-s} * f_\varepsilon(x)\, p_{t'-s'+t'-s} * f_\varepsilon(x)\, \mathrm{d}x \ldots$$

$$= |\mu| \int p_{s-r+s'-r}(x)\, p_{t-s'+t-s} * f_\varepsilon(x)\, p_{t'-s'+t'-s} * f_\varepsilon(x)\, \mathrm{d}x \ldots$$

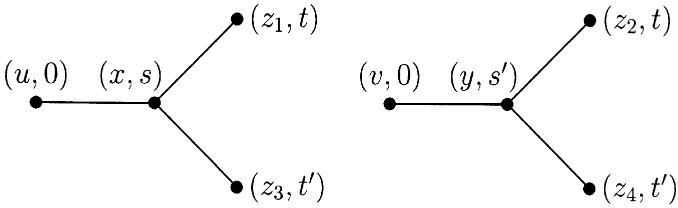
$$= 2|\mu| \int p_{2s+(s'-s)}(x)\, p_{2(t-s)+(s'-s)} * f_\varepsilon(x)\, p_{2(t'-s)+(s'-s)} * f_\varepsilon(x)\, \mathrm{d}x \ldots$$

$$= \frac{1}{4}|\mu| \int G * p_v(x) (G * p_v * f_\varepsilon(x))^2\, \mathrm{d}x\, \mathrm{d}v, \tag{3.2}$$



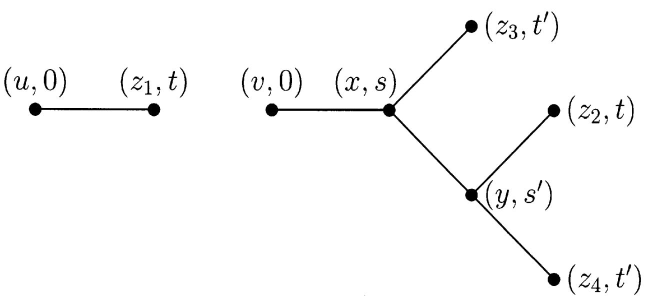
Graph 2

$$\int \mu(\mathrm{d}u)p_r(z-u)p_{t-r}(z_1-z)p_{s'-r}(y-z)p_{s-s'}(x-y)p_{t-s}(x-z_2)p_{t'-s}(x-z_3)$$
$$p_{t'-s'}(y-z_4)f_\varepsilon(z_1-z_2)f_\varepsilon(z_3-z_4)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\bar{z}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}t\,\mathrm{d}t'$$
$$=|\mu|\int p_{s'}(y)p_{s-s'}(x-y)p_{2(t'-s)+s-s'}*f_\varepsilon(x-y)p_{t-s+t}*f_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$=|\mu|\int p_{s'}(y)p_{s-s'}(x)p_{2(t'-s)+s-s'}*f_\varepsilon(x)p_{t-s+t}*f_\varepsilon(x+y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=|\mu|\int p_{s-s'}(x)p_{2(t'-s)+s-s'}*f_\varepsilon(x)p_{t-s+t+s'}*f_\varepsilon(x)\,\mathrm{d}x$$
$$=|\mu|\int p_{s-s'}(x)p_{2(t'-s)+(s-s')}*f_\varepsilon(x)p_{2(t-s)+2s'+(s-s')}*f_\varepsilon(x)\,\mathrm{d}x$$
$$=\frac{1}{8}|\mu|\int p_v(x)G*p_v*f_\varepsilon(x)G*G*p_v*f_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}v, \tag{3.3}$$



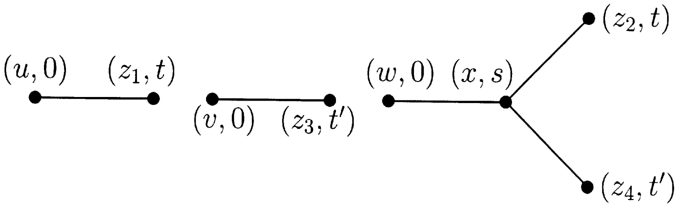
Graph 3

$$\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)p_s(u-x)p_{s'}(v-y)p_{t-s}(x-z_1)p_{t'-s}(x-z_3)p_{t-s'}(y-z_2)$$
$$\times p_{t'-s'}(y-z_4)f_\varepsilon(z_1-z_2)f_\varepsilon(z_3-z_4)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\bar{z}\,\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}t\,\mathrm{d}t'$$
$$=\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)p_s(u-x)p_{s'}(v-y)p_{t-s+t-s'}*f_\varepsilon(x-y)p_{t'-s+t'-s'}$$
$$*f_\varepsilon(x-y)\,\mathrm{d}x\ldots$$
$$=\frac{1}{4}\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)G*p_r(x-(u-v))(G*p_r*f_\varepsilon(x))^2\,\mathrm{d}x\,\mathrm{d}r, \tag{3.4}$$



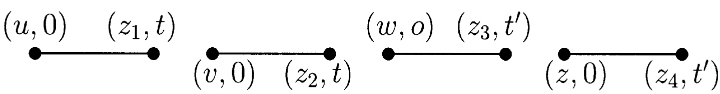
Graph 4

$$\begin{aligned}
&\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)\,p_t(u-z_1)\,p_s(v-x)\,p_{t'-s}(x-z_3)\,p_{s'-s}(x-y)\,p_{t-s'}(y-z_2)\\
&\quad\times p_{t'-s'}(y-z_4)\,f_\varepsilon(z_1-z_2)\,f_\varepsilon(z_3-z_4)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\bar{z}\,\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}t\,\mathrm{d}t'\\
&\quad=\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)\,p_s(v-x)\,p_{t-s'+t}*f_\varepsilon(u-y)\,p_{s'-s}(x-y)\,p_{t'-s+t'-s'}\\
&\qquad * f_\varepsilon(x-y)\,\mathrm{d}x\ldots\\
&\quad=\frac{1}{8}\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)G*G*p_r*f_\varepsilon(x-(u-v))\,p_r(x)G*p_r*f_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}r,
\end{aligned} \tag{3.5}$$



Graph 5

$$\begin{aligned}
&\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)\mu(\mathrm{d}w)\,p_s(x-w)\,p_{t-s+t}*f_\varepsilon(x-u)\,p_{t'-s+t'}*f_\varepsilon(x-v)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}t'\\
&\quad=\frac{1}{4}\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)\mu(\mathrm{d}w)\,p_s(x-w)G*p_s*f_\varepsilon(x-u)G*p_s*f_\varepsilon(x-v)\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{3.6}$$



Graph 6

$$\frac{1}{4}\int \mu(\mathrm{d}u)\mu(\mathrm{d}v)\mu(\mathrm{d}w)\mu(\mathrm{d}z)G*f_\varepsilon(u-v)G*f_\varepsilon(w-z). \tag{3.7}$$

Let $u_\alpha(x)$ denote a generic measurable function which falls off exponentially and monotonically in $|x|$, and such that $|x|\to\infty$, and

$$|u(x)|\leqslant c\frac{1}{|x|^\alpha},$$

and let $u_{\alpha,\varepsilon}(x)$ denote a generic measurable function which falls off exponentially and monotonically in $|x|$, and such that

$$\begin{aligned}
&u_{\alpha,\varepsilon}(x)\leqslant c|x|^{-\alpha},\quad |x|\geqslant\varepsilon,\\
&u_{\alpha,\varepsilon}(x)\leqslant c\varepsilon^{-\alpha},\quad |x|\leqslant\varepsilon.
\end{aligned}$$

With $u_{o,\varepsilon}$ we associate $\lg(1/|x|)$ instead of $|x|^{-\alpha}$.

We will use the following simple lemma from Rosen (1992):

Lemma 1. *If* $\alpha<d$ *then* $u_\alpha * f_\varepsilon(x)$ *has exponential falloff as* $|x|\to\infty$, *and*

$$|u_\alpha * f_\varepsilon(x)| \leqslant \begin{cases} \bar{c}\cdot\dfrac{1}{|x|^\alpha}, & |x|\geqslant\varepsilon, \\ \bar{c}\cdot\dfrac{1}{\varepsilon^\alpha}, & |x|\leqslant\varepsilon, \end{cases} \tag{3.8}$$

i.e. $u_\alpha * f_\varepsilon(x)=u_{\alpha,\varepsilon}(x)$.

The functions $G, G*G$ are of the above form as we saw in Section 2. They all have exponential falloff as $|x|\to\infty$, while for small x we have the bounds:

$$\begin{array}{llll} & \underline{d=2} & \underline{d=3} & \underline{d=4} \\ G(x) & c\lg(\frac{1}{|x|}) & cx^{-1} & cx^{-2} \\ G*G(x) & c & c & c\lg\left(\frac{1}{|x|}\right) \end{array} \tag{3.9}$$

Using Eq. (3.9) and Lemma 4, it is easy to check that all the integrals in formulas (3.2)–(3.7) are uniformly bounded as $\varepsilon\to 0$ when $d=2$ or 3.

We thus concentrate on $d=4$. The integrals for graphs 3,5 and 6 are uniformly bounded as $\varepsilon\to 0$, while the above shows that the integral for graph 4 is $O(\lg(1/\varepsilon))$. To obtain a similar bound on the integral for Graph 1 we first note that

$$\int\prod_{i=1}^{3} p_v(y_i)\,\mathrm{d}v \leqslant \prod_{i=1}^{3}\left(\int p_v^3(y_i)\,\mathrm{d}v\right)^{1/3} = \prod_{i=1}^{3} u_{10/3}(y_i)$$

and that $u_{10/3} * u_2(y)=u_{4/3}(y)$ hence

$$\begin{aligned} &\frac{1}{4}|\mu|\int G*p_v(x)(G*p_v*f_\varepsilon(x))^2\mathrm{d}x\,\mathrm{d}v \\ &\leqslant c\int u_{4/3}(x)u^2_{4/3,\varepsilon}(x)\,\mathrm{d}x \\ &\leqslant c\lg\left(\frac{1}{\varepsilon}\right). \end{aligned} \tag{3.10}$$

We now carefully compute the integral (3.3) corresponding to Graph 2. We will show that it is $\sim c(\lg(1/\varepsilon))^2$.

Using Eqs. (2.18) and (2.17) we first obtain an upper bound:

$$\begin{aligned} J(\varepsilon) &= \frac{1}{8}\int p_v(x)G*p_v*f_\varepsilon(x)G*G*p_v*f_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}v \\ &\leqslant \frac{1}{8}\int G(x)G*f_\varepsilon(x)\,G*G*f_\varepsilon(x)\,\mathrm{d}x \\ &= \frac{1}{8}\int_{|x|\leqslant 1/2} G(x)G*f_\varepsilon(x)G*G*f_\varepsilon(x)\,\mathrm{d}x+0(1) \\ &= \frac{1}{8}\int_{|x|\leqslant 1/2}\frac{1}{2\pi^2}x^{-2}\frac{1}{2\pi^2}(x^{-2}*f_\varepsilon)(x) \end{aligned}$$

$$\frac{1}{2\pi^2}\left(\lg\left(\frac{1}{|x|}\right)*f_\varepsilon\right)(x)+0\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^6}\int_{2\varepsilon\leqslant|x|\leqslant 1/2} x^{-2}(x^{-2}*f_\varepsilon)(x)$$

$$\left(\lg\left(\frac{1}{|x|}\right)*f_\varepsilon\right)(x)+0\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^6}\int_{2\leqslant|x|\leqslant 1/2\varepsilon}\int\frac{1}{x^2}\frac{1}{|x-z|^2}\lg\left(\frac{1}{\varepsilon|x-y|}\right)f(y)f(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+0\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^6}\int_{2\leqslant|x|\leqslant 1/2\varepsilon}\frac{1}{x^4}\left(\lg\left(\frac{1}{\varepsilon}\right)+\lg\left(\frac{1}{|x|}\right)\right)\mathrm{d}x+\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^6}2\pi^2\int_2^{1/2\varepsilon}\frac{1}{r}\left(\lg\left(\frac{1}{\varepsilon}\right)-\lg(r)\right)\mathrm{d}r+\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right)+\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right). \tag{3.11}$$

We now obtain a lower bound whose leading term is the same as that obtained in the upper bound.

$$J(\varepsilon) = \frac{1}{8}\int\int_0^\infty p_v(x)G*p_v*f_\varepsilon(x)G*G*p_v*f_\varepsilon(x)\,\mathrm{d}v\,\mathrm{d}x$$

$$\geqslant \frac{1}{8}\int\int_0^{\varepsilon^2} p_v(x)G*p_v*f_\varepsilon(x)G*G*p_v*f_\varepsilon(x)\,\mathrm{d}v\,\mathrm{d}x$$

$$\geqslant \frac{1}{8}\int G*p_{\varepsilon^2}*f_\varepsilon(x)\,G*G*p_{\varepsilon^2}*f_\varepsilon(x)\left(\int_0^{\varepsilon^2}p_v(x)\,\mathrm{d}v\right)\mathrm{d}x$$

$$= \frac{1}{8}\int_{|x|\leqslant 1/2} G*p_{\varepsilon^2}*f_\varepsilon(x)G*G*p_{\varepsilon^2}*f_\varepsilon(x)\left(\int_0^{\varepsilon^2}p_v(x)\,\mathrm{d}v\right)\mathrm{d}x-\mathrm{O}(1)$$

$$= \frac{1}{8}\int_{2\varepsilon\leqslant|x|\leqslant 1/2}\frac{1}{2\pi^2|x|^2}*p_{\varepsilon^2}*f_\varepsilon(x)\frac{1}{2\pi^2}\lg\left(\frac{1}{|x|}\right)$$

$$*\,p_{\varepsilon^2}*f_\varepsilon(x)\left(\int_0^{\varepsilon^2}p_v(x)\,\mathrm{d}v\right)\mathrm{d}x-\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{32\pi^4}\int_{2\leqslant|x|\leqslant 1/2\varepsilon}|x|^{-2}*p_1*f(x)\lg\left(\frac{1}{\varepsilon|x|}\right)$$

$$*\,p_1*f(x)\left(\varepsilon^2\int_0^{\varepsilon^2}p_v(\varepsilon x)\,\mathrm{d}v\right)\mathrm{d}x-\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^6}\int_{2\leqslant|x|\leqslant 1/2\varepsilon} |x|^{-2} * p_1 * f(x)\,\lg\left(\frac{1}{\varepsilon|x|}\right) * p_1 * f(x)\frac{1}{|x|^2}\,\mathrm{d}x - \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{1}{64\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right) - \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right) \tag{3.12}$$

as in the calculation of Eq. (3.11). Here we have used

$$\varepsilon^2\int_0^{\varepsilon^2} p_v(\varepsilon x)\,\mathrm{d}v$$
$$= \int_0^1 \mathrm{e}^{-\varepsilon^2 t}\frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t$$
$$= \int_0^1 \frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t - \int_0^1 (1-\mathrm{e}^{-\varepsilon^2 t})\frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t$$
$$= G_0(x) - \int_1^\infty \frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t - \mathrm{O}\left(\varepsilon^2\int_0^1 \frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t\right)$$
$$= \{G_0(x) - \mathrm{O}(\mathrm{e}^{-x^2/2})\}(1+\mathrm{O}(\varepsilon^2))$$

for $2\leqslant|x|\leqslant 1/2\varepsilon$.

It can be shown as in Rosen (1992) that the number of graphs in C_4 which give rise to a contribution (3.3) is precisely 4^3.

Thus

$$I(\varepsilon) = 4^3|\mu|J(\varepsilon) + \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right) = \frac{1}{\pi^4}|\mu|\lg^2\left(\frac{1}{\varepsilon}\right) + \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right). \tag{3.13}$$

4. Proof of Theorem 2(a) and (b)

Let

$$\gamma_\varepsilon = \int_0^\infty \langle f_\varepsilon(x-y),\ X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s, \tag{4.1}$$

where

$$c(\varepsilon) = \begin{cases} \dfrac{1}{\pi}\lg\left(\dfrac{1}{\varepsilon}\right), & d=2,\\[2ex] \dfrac{1}{2\pi}\dfrac{1}{\varepsilon}\displaystyle\int \frac{f(y)}{|y|}\,\mathrm{d}y, & d=3. \end{cases}$$

Almost precisely as in Rosen (1992) we can show that

$$E_\mu[(\gamma_\varepsilon - \gamma_{\bar\varepsilon})^2] \to 0 \quad \text{as } \varepsilon, \bar\varepsilon \to 0 \tag{4.2}$$

and this completes the proof of Theorem 2(a) and (b).

5. Proof of Theorem 2(c): Combinatorial aspect

Our proof is by the method of moments.

Recall that

$$\gamma_\varepsilon = \int_0^\infty \langle f_\varepsilon(x-y),\ X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s - c(\varepsilon)\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s, \tag{5.1}$$

where

$$\begin{aligned} c(\varepsilon) &= \int f_\varepsilon(x)G(x)\,\mathrm{d}x \\ &= \frac{1}{2\pi^2}\frac{1}{\varepsilon^2}\int \frac{f(y)}{y^2}\mathrm{d}y - \frac{1}{2\pi^2}\lg\left(\frac{1}{\varepsilon}\right) + 0(1). \end{aligned} \tag{5.2}$$

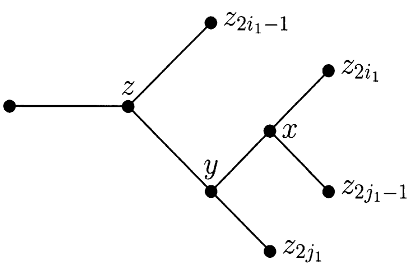
By Eq. (2.10) we know that

$$E_\mu(\gamma_\varepsilon^{2m}) \tag{5.3}$$

is a sum of contributions from the graphs of C_{4m}, i.e. the set of binary graphs with $4m$ labeled exits, $1,2,\dots,4m$ with no twin exits coupled – i.e. no twin exits are ever labeled $2i-1, 2i$ for any i.

The basic idea which we explain in this and the next section is that the dominant contribution to Eq. (5.3) comes from graphs which effectively break Eq. (5.3) up into a product of m second moments.

Let $A_{4m} \subset C_{4m}$ denote those binary graphs in C_{4m} for which there is a complete pairing $(i_1, j_1), \dots, (i_m, j_m)$ of the $2m$ integers $1,2,\dots,2m$ and such that for each such pair (i_ℓ, j_ℓ) the exits labeled $2i_\ell - 1, 2i_\ell,\ 2j_\ell - 1,\ 2j_\ell$ are arranged as in Graph 2 of Section 3:



(5.4)

or one of its 4^3 variants as described at the end of Section 3.

We will see later that the dominant contribution to Eq. (5.3) comes from the graphs in A_{4m}, and is of order $\lg^{2m}(1/\varepsilon)$, while any other graph in C_{4m} will give a contribution which is $\mathrm{O}(\lg^{2m-1}(1/\varepsilon))$.

Let us compute the contribution from the graphs in A_{4m}. Consider the subgraph (5.4). The partial integral with respect to $\mathrm{d}x\mathrm{d}y\ \mathrm{d}z_{2i_{\ell-1}}\ \mathrm{d}z_{2i_\ell}\ \mathrm{d}z_{2j_{\ell-1}}\ \mathrm{d}z_{2j_{\ell-1}}$ is described in Eq. (3.3). It is crucial that this partial integral is independent of z and r (a consequence of the translation invariance of Brownian motion), and is simply the constant (see Eq. (3.11))

$$J(\varepsilon) = \frac{1}{4^3\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right) + \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right). \tag{5.5}$$

As we saw at the end of Section 3, there are 4^3 variants of Eq. (5.4). Thus the partial integration corresponding to all m pairs (i_ℓ, j_ℓ) and all the 4^3 variants for each pair gives rise to the factor

$$\left(\frac{1}{\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right)\right)^m + \mathrm{O}\left(\lg^{2m-1}\left(\frac{1}{\varepsilon}\right)\right). \tag{5.6}$$

After this partial integration, we are simply left with a binary graph with m exits. Since any graph in D_m can arise in this fashion, and since there are $(2m)!/m!2^m$ ways to pair the integers $1,2,\dots,2m$ we see that (see Eq. (2.4)) the contribution to Eq. (5.3) from A_{4m} is

$$\frac{(2m)!}{m!2^m}\left(\frac{1}{\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right)\right)^m E_\mu\left(\left(\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s\right)^m\right) + \mathrm{O}\left(\lg^{2m-1}\left(\frac{1}{\varepsilon}\right)\right). \tag{5.7}$$

We will show in the next section that the contribution of all graphs in $C_{4m} - A_{4m}$ is $\mathrm{O}(\lg^{2m-1}(1/\varepsilon))$. This will give

$$E_\mu\left[\left(\frac{\gamma_\varepsilon}{\lg(1/\varepsilon)}\right)^{2m}\right] \longrightarrow \frac{(2m)!}{m!}\left(\frac{1}{2\pi^4}\right)^m E_\mu\left(\left(\int_0^\infty \langle 1, X_s\rangle\,\mathrm{d}s\right)^m\right) \quad \text{as } \varepsilon \to 0. \tag{5.8}$$

Furthermore, the next section will show that

$$E_\mu\left[\left(\frac{\gamma_\varepsilon}{\lg(1/\varepsilon)}\right)^{2m-1}\right] \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \tag{5.9}$$

Let M_{2m} denote the right-hand side of Eq. (5.8). A simple combinatoric argument spelled out in Rosen (1992) shows that for $|\lambda|$ small,

$$\sum_{m=0}^\infty \frac{\lambda^{2m} M_{2m}}{(2m)!} = \mathrm{e}^{|\mu|(1/2)(1-\sqrt{1-2\lambda^2/\pi^4})}. \tag{5.10}$$

This shows at once that any limit distribution of $\gamma_\varepsilon/\lg(1/\varepsilon)$ is determined by its moments, hence unique, and also shows that its Laplace transform is given by Eq. (5.10), which establishes Theorem 2(c).

6. Proof of Theorem 2(c): analytic aspect

We recall from Eq. (2.10) that

$$E_\mu(\gamma_\varepsilon^n) = \sum_{C_{2n}} \int \prod_{v\in V_-} \mu(\mathrm{d}y_v) \prod_{a\in A} p_{s_{f(a)}-s_{i(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v\in V_0} \mathrm{d}s_v \mathrm{d}y_v$$
$$\times \prod_{i=1}^n f_\varepsilon(z_{2i} - z_{2i-1})\,\mathrm{d}z_{2i}\,\mathrm{d}z_{2v}\,\mathrm{d}r_i. \tag{6.1}$$

In this section, we show that unless $n = 2m$ and the graph C is in A_{4m}, then the contribution of C to Eq. (6.1) is

$$0\left(\lg^{n-1}\left(\frac{1}{\varepsilon}\right)\right). \tag{6.2}$$

As discussed in Section 5, this will complete the proof of Theorem 2(c).

We can think of the integral in Eq. (6.1) as obtained by assigning a factor $p_{s_{f(a)}-s_{i(a)}}(y_{f(a)}-y_{i(a)})$ to each arrow $a\in A$. We must integrate out all internal variables $\mathrm{d}y_v$, $v\in V_o$, all entrances with respect to $\mathrm{d}\mu$, all exits with $\prod_{i=1}^n f_\varepsilon(z_{2i}-z_{2i-1})\,\mathrm{d}z_{2i-1}\,\mathrm{d}z_{2i}$, as well as all time variables.

Our approach to Eq. (6.2) is to successively integrate out the variables, at each stage replacing the graph C by a different graph C' (not necessarily a directed or binary graph).

The arrows of C' are associated with factors described below, such that the contribution of C is bounded by that of C'. In this process we will be able to associate a factor $0(\lg(1/\varepsilon))$ to each f_ε in Eq. (6.1) in such a way that these factors will bound all divergences as $\varepsilon\to 0$, and we will show that unless $n=2m$ and $C\subseteq A_{4m}$, at least one of the factors associated to some f_ε will be $0(1)$.

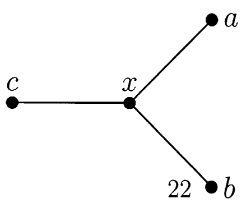
Here are the details:

We begin by integrating the exit variables $z_1,\ldots,z_{2n}$. We obtain n factors of the form

$$\int p_{t_i-\cdot}(a-z_{2i-1})f_\varepsilon(z_{2i}-z_{2i-1})p_{t_i-\cdot}(b-z_{2i})\,\mathrm{d}z_{2i-1}\,\mathrm{d}z_{2i}\leqslant cG*f_\varepsilon(b-a). \qquad (6.3)$$

We know from the fact that $C\subseteq C_{2m}$, that $a\not\equiv b$. Form a new graph C' obtained by putting an edge between $i(u)$ and $i(v)$ whenever $f(u)=z_{2i-1},\ f(v)=z_{2i}$, i.e. we connect the vertices associated with a,b in Eq. (6.3). With this new edge, called a 'leading edge', we associate the factor $G*f_\varepsilon$.

Assume that C' has a subgraph of the form



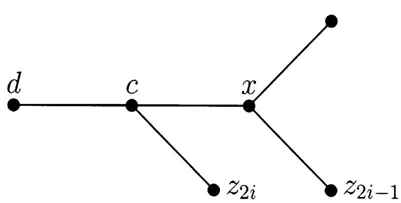
(6.4)

where $(x,a),(x,b)$ are both leading edges. We distinguish three possibilities:

1. $a\equiv c$, or $b\equiv c$ (We cannot have both.)
2. $a\equiv b$
3. a,b,c are distinct.

We analyze each in turn:

(i) Assume that $b\equiv c$. This can only have occurred if C contained the subgraph



(6.5)

Since we think of z_{2i}, z_{2i-1} as connected by f_ε, we refer to the situation in Eq. (6.5) as a simple loop.

The partial integral over x in this case is bounded by

$$\int G(c-x)G*f_\varepsilon(c-x)G*f_\varepsilon(a-x)\,\mathrm{d}x = \int G(x)G*f_\varepsilon(x)G*f_\varepsilon(a-c-x)\,\mathrm{d}x. \tag{6.6}$$

We know from Lemma 1 that

$$G*f_\varepsilon \leqslant u_{2,\varepsilon}.$$

If $|x| \geqslant \frac{1}{2}|a-c|$, Eq. (6.6) is bounded by

$$u_{2,\varepsilon}(a-c)\int G(x)G*f_\varepsilon(a-c-x)\,\mathrm{d}x = u_{2,\varepsilon}(a-c)u_{0,\varepsilon}(a-c) \tag{6.7}$$

as we see from Eq. (3.9). While if $|x| \leqslant \frac{1}{2}|a-c|$, so that $|a-c-x| \geqslant (|a-c|)/2$, Eq. (6.6) is bounded by

$$u_{2,\varepsilon}(a-c)\int G(x)G*f_\varepsilon(x)\,\mathrm{d}x = \lg\left(\frac{1}{\varepsilon}\right)u_{2,\varepsilon}(a-c). \tag{6.8}$$

In any event, Eq. (6.6) is bounded by $\lg(1/\varepsilon)u_{2,\varepsilon}(a-c)$. (It is important to recall that we cannot have $a \equiv c$.) We then form a new graph C'', with an edge between the vertices associated with a and c. We consider the factor $\lg(1/\varepsilon)$ as associated with $f_\varepsilon(z_{2i}-z_{2i-1})$, and associate $u_{2,\varepsilon}$ to our new edge, now called a leading edge.

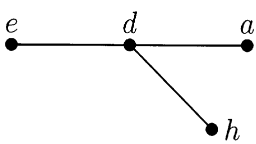
Because Eq. (6.5) refers to a binary graph, in C'', aside from our new edge connecting a and c, there is only one other arrow connecting c, with a factor $G(c-d)$. We now integrate

$$\int G(c-d)u_{2,\varepsilon}(a-c)\,\mathrm{d}c = u_{0,\varepsilon}(d-a). \tag{6.9}$$

(This integral was already computed in Eq. (6.7).)

If $a \equiv d$, we are in the situation of Section 3, Graph 2, i.e. our subgraph (6.5) was precisely of the form making up A_{4m}, contributing $\lg^2(1/\varepsilon)$ which we associate with the two f_ε factors for that subgraph, which now have no further influence.

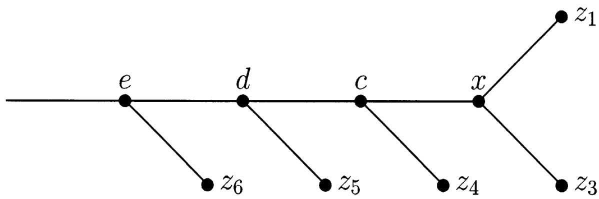
If $a \not\equiv d$, we form a new graph C'' linking a and d, and with the factor $u_{0,\varepsilon}(d-a)$. We have a subgraph



(6.10)

which looks like Eq. (6.4), except that instead of the factor $u_{2,\varepsilon}$ associated to (d,a), we have $u_{0,\varepsilon}$.

We will see after analyzing cases (ii) and (iii) that the worst possible case comes from the two loop subgraph



(6.11)

i.e. $h \equiv e$ (so that $a \not\equiv e$) and the partial integral over d is bounded by

$$\int G(e-d)u_{2,\varepsilon}(e-d)u_{0,\varepsilon}(d-a)d(d) = \int G(x)u_{2,\varepsilon}(x)u_{0,\varepsilon}(a-e-x)\,\mathrm{d}x \tag{6.12}$$

with $a \not\equiv e$.

As in the analysis of Eq. (6.6) we find Eq. (6.12) bounded by

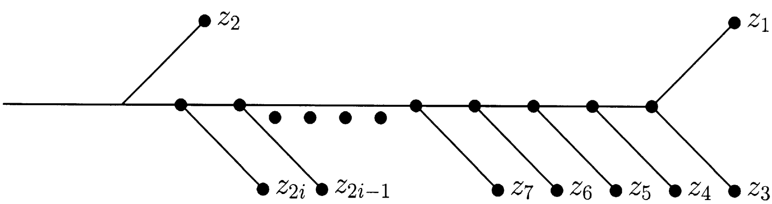
$$\lg\left(\frac{1}{\varepsilon}\right) u_{\mathrm{O},\varepsilon}(a-e) + u_{2,\varepsilon}(a-e) \tag{6.13}$$

so that the de integral is

$$\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right). \tag{6.14}$$

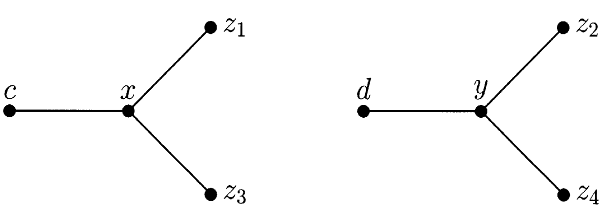
Thus, three factors of f_ε give rise only to a $\lg^2(1/\varepsilon)$ contribution (as opposed to $\lg^3(1/\varepsilon)$).

In particular, a subgraph of the form



with $i \geqslant 2$ loops, gives a contribution which is $\mathrm{O}(\lg^{i-1}(1/\varepsilon))$, unless $i = 2$.

(ii) This case arises from the subgraph



The partial integral with respect to $z_1,\dots,z_4,x,y$ is bounded by

$$\begin{aligned}
&\iint p_r(x-c)p_{t-r+t-s}*f_\varepsilon(x-y)p_{t'-r+t'-s}*f_\varepsilon(x-y)p_s(y-d)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}t' \\
&\quad=\int p_{r+s}(c-d-x)p_{t-r+t-s}*f_\varepsilon(x)p_{t'-r+t'-s}*f_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}t' \\
&\quad=\frac{2}{8}\int G*p_v(c-d-x)(G*p_v*f_\varepsilon(x))^2\,\mathrm{d}x\,\mathrm{d}v \\
&\quad\leqslant u_{0,\varepsilon}(c-d)
\end{aligned}\tag{6.15}$$

as in Eq. (6.7) after using Holder's inequality in the dv integral as in Eq. (3.10).

If $c\equiv d$ (which is the situation of Section 3, Graph 1), we have a $\lg(1/\varepsilon)$ for two factors of f_ε, while if $c\not\equiv d$ we can also bound

$$u_{0,\varepsilon}(c-d)\leqslant\lg\left(\frac{1}{\varepsilon}\right)u(c-d),$$

where $u(c-d)$ is bounded, and falls off exponentially as $|c-d|\to\infty$. We have a factor $\lg(1/\varepsilon)$ for the two f_ε's, and a new graph with an edge connecting the vertices associated with c and d, and associated factor $u(c-d)$.

(iii) If a,b,c are distinct, the partial x integral is

$$\int G(c-x)G*f_\varepsilon(x-a)G*f_\varepsilon(x-b)\,\mathrm{d}x.\tag{6.16}$$

If the variable a or b no longer appears in any other factors associated with edges of our graph – we perform the da or db integral. If, e.g., we first do the da integration, then Eq. (6.16) is bounded by

$$u_{0,\varepsilon}(c-b)\leqslant\lg\left(\frac{1}{\varepsilon}\right)u(c-b)\tag{6.17}$$

and as in the discussion of (ii), we associate $\lg(1/\varepsilon)$ with two f_ε factors.

If both a and b appear in other factors, we use

$$uv\leqslant\tfrac{1}{2}(u^2+v^2)$$

to bound Eq. (6.16) by

$$G*u_{4,\varepsilon}(a-c)+G*u_{4,\varepsilon}(b-c)\leqslant\lg\left(\frac{1}{\varepsilon}\right)(u_{2,\varepsilon}(a-c)+u_{2,\varepsilon}(b-c)).\tag{6.18}$$

We now form two new graphs, one with a new edge connecting a and c, with a factor $u_{2,\varepsilon}(a-c)$ – and analogously for the other graph.

It suffices to consider the first graph. Notice that the factor $u_{2,\varepsilon}(a-c)$ is the type of factor we obtained from the initial integration over exits – hence we can continue our analysis as if it arose in the latter manner – with the difference that we have actually used up two f_ε factors at the cost of one $\lg(1/\varepsilon)$ factor. This could only lead to problems if our new $u_{2,\varepsilon}(a-c)$ were part of a two-loop graph, and it is easily seen that that is impossible because of the dx integration in Eq. (6.16).

We can now return to the end of our discussion of case (i), and see that indeed the worst possible case for Eq. (6.10) is as described there – i.e. Eq. (6.11).

By iterating (i)–(iii), applied to leading edges, we see that Eq. (6.2) holds.

7. Proof of Theorem 1

In analogy with Eq. (2.10) we find

$$E_\mu(\gamma_\varepsilon^n(T)) = \sum_{C_{2n}} \int \prod_{v\in V_-} \mu(\mathrm{d}y_v) \prod_{a\in A} p_{s_{f(a)}-s_{i(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v\in V_o} \mathrm{d}s_v\, \mathrm{d}y_v$$
$$\times 1_{\{r_j\leqslant T,\forall_j\}} \prod_{i=1}^n f_\varepsilon(z_{2i-1} - z_{2i})\, \mathrm{d}z_{2i-1}\, \mathrm{d}z_{2i}\, \mathrm{d}r_i \tag{7.1}$$

where now

$$p_s(y) = \frac{\mathrm{e}^{-y^2/2t}}{(2\pi t)^{d/2}}$$

is the transition density for Brownian motion in $\mathbb{R}^d$.

Note that by inserting factors $\mathrm{e}^{-(s_{f(a)}-s_{i(a)})} \geqslant \mathrm{e}^{-T}$, we can bound the contribution to Eq. (7.1) of any graph C, by its contribution to Eq. (2.10). This immediately shows that if $d=2,3$ and $n=2$, then Eq. (7.1) is uniformly bounded in ε, while if $d=4$, we can bound the contribution of each graph to Eq. (7.1) by $c\,\mathrm{lg}^n(1/\varepsilon)$, and in fact, unless $n=2m$ and our graph belongs to A_{4m}, then its contribution can be bounded by $c\,\mathrm{lg}^{n-1}(1/\varepsilon)$.

The L^2 convergence for $d=2,3$ follows easily by using such a domination together with Eq. (4.2). The case of $d=4$ is more subtle.

We consider in detail the contribution of a subgraph of the type described by Graph 2 of Section 3. This contribution is

$$J(r,\varepsilon) \doteq |\mu| \int \int_A p_{s-s'}(x) p_{2(t'-s)+(s-s')} * f_\varepsilon(x) p_{2(t-s)+2(s'-r)+(s-s')}$$
$$* f_\varepsilon(x)\, \mathrm{d}s'\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}x$$
$$= |\mu| \int \int_A p_{s-s'}(x) p_{2(t'-s)} * p_{s-s'} * f_\varepsilon(x) p_{2(t-s)} * p_{2(s'-r)} * p_{s-s'}$$
$$* f_\varepsilon(x)\, \mathrm{d}s'\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}x$$

where $A = \{(s',s,t,t') \mid r \leqslant s' \leqslant s \leqslant t, t' \leqslant T\}$.

Recall from Eqs. (3.11) and (3.12) that

$$J(\varepsilon) = \frac{1}{8} \int p_v(x) G * p_v * f_\varepsilon(x) G * G * p_v * f_\varepsilon(x)\, \mathrm{d}x\, \mathrm{d}v$$
$$= \frac{1}{64\pi^4} \mathrm{lg}^2\left(\frac{1}{\varepsilon}\right) + 0\left(\mathrm{lg}\left(\frac{1}{\varepsilon}\right)\right). \tag{7.2}$$

We now show that for any fixed $\delta > 0$,

$$J(r,\varepsilon) = J(\varepsilon) + 0\left(\mathrm{lg}\left(\frac{1}{\varepsilon}\right)\right), \quad T - r \geqslant 3\delta \tag{7.3}$$

Let

$$q_t = \mathrm{e}^{-t} p_t(x), \qquad G^r(x) = \int_0^r \mathrm{e}^{-t} p_t(x)\,\mathrm{d}t.$$

Using

$$|1 - \mathrm{e}^{-t}| \leqslant 2t$$

we easily check that

$$J(r,\varepsilon) = |\mu| \int\!\!\int_A q_{s-s'}(x) q_{2(t'-s)} * p_{s-s'} * f_\varepsilon(x) q_{2(t-s)} * q_{2(s'-r)} * p_{s-s'} \\ * f_\varepsilon(x)\,\mathrm{d}s'\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}x + 0\left(\lg\left(\frac{1}{\varepsilon}\right)\right). \tag{7.4}$$

Note that under our assumption that $T - r \geqslant 3\delta$ we have that

$$A = \{(s', s, t, t') \mid r \leqslant s' \leqslant s \leqslant t, t' \leqslant T\} \\ \supseteq B \doteq \{(s', s, t, t') \mid 0 \leqslant s' - r \leqslant \delta, 0 \leqslant s - s' \leqslant \delta, 0 \leqslant t - s \leqslant \delta, 0 \leqslant t' - s \leqslant \delta, \}.$$

Using the bound

$$q_r(x) \leqslant \frac{c\mathrm{e}^{-x^2/2r}}{\delta^2}$$

if

$$r \geqslant \delta,$$

we see that in Eq. (7.4) we can assume that the integral is over the region B and using the bound

$$|G(x) - G^r(x)| \leqslant c \int_r^\infty \frac{\mathrm{e}^{-x^2/2t}}{t^2}\,\mathrm{d}t = u(x)$$

and the methods used to obtain Eq. (7.2) we see that

$$J(r,\varepsilon) = \frac{1}{8} |\mu| \int\!\!\int_0^\delta q_v(x) G * p_v * f_\varepsilon(x) G * G * p_v * f_\varepsilon(x)\,\mathrm{d}v\,\mathrm{d}x + 0\left(\lg\left(\frac{1}{\varepsilon}\right)\right) \\ = J(\varepsilon) + 0\left(\lg\left(\frac{1}{\varepsilon}\right)\right) \tag{7.5}$$

which proves Eq. (7.3).

The rest of the proof now follows as in Rosen (1992).

8. Theorem 3: superprocesses over diffusions

Let z_t to a smooth uniformly elliptic diffusion in $\mathbb{R}^d$, with transition density $p_s(x, y)$. It is easy to write down the analogue of Eq. (7.1) for Z_t, the superprocess over z_t: simply replace

$$p_{s_{f(a)} - s_{i(a)}}(y_{f(a)} - y_{i(a)})$$

by

$$p_{s_{f(a)}-s_{i(a)}}(y_{i(a)}, y_{f(a)}).$$

Since, for some $M, \alpha > 0$

$$p_s(x,y) \leqslant M \frac{e^{-\alpha(x-y)^2/2t}}{(2\pi t)^{d/2}}, \tag{8.1}$$

we can apply all the results of the previous sections to obtain bounds on the moments of $\gamma_\varepsilon(T)$.

In particular, if $d=2$ or 3, $E_\mu(\gamma_\varepsilon^2(T))$ is uniformly bounded in ε, and convergence in L^2 follows using Rosen (1987), (2.4), (3.16)).

When $d=4$, the same reasoning shows that we can bound the contribution of any graph C to $E_\mu(\gamma_\varepsilon^n(T)$ by $\lg^{n-1}(1/\varepsilon)$ unless $n=2m$ and $C \subseteq A_{4m}$.

As in the previous section, it suffices to show that

$$\begin{aligned}
&\int \mu(\mathrm{d}u) p_r(z,u) p_{t-r}(z_1,z) p_{s'-r}(y,z) p_{s-s'}(x,y) p_{t-s}(x,z_2) p_{t'-s}(x,z_3) p_{t'-s'}(y,z_4) \\
&\quad f_\varepsilon(z_1-z_2) f_\varepsilon(z_3-z_4)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\bar{z}\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}s'\, \mathrm{d}t\, \mathrm{d}t' \\
&\quad = |\mu| \frac{1}{64\pi^4} \lg^2\left(\frac{1}{\varepsilon}\right) + 0\left(\lg\left(\frac{1}{\varepsilon}\right)\right).
\end{aligned} \tag{8.2}$$

However, using the bounds just described, we know that up to errors of order $\lg(1/\varepsilon)$, we can restrict integration to the region where $z_1, x, y, z_1, \ldots, z_4$ are close together. It is known that for x near y

$$p_t(x,y) = \frac{e^{-|C^{-1}(x-y)|^2/2t}}{(2\pi t)^{d/2} \det(C)} + 0\left(\frac{1}{t^{d/2-\beta}} e^{-\alpha|x-y|^2/2t}\right) \tag{8.3}$$

for some $\beta > 0$, where $C = \sqrt{A(x)}$ and $A(x)$ in the matrix $a_{ij}(x)$.

We thus see that up to errors $0(\lg(1/\varepsilon))$, the integral in Eq. (8.2) is equal to

$$\begin{aligned}
&\int \mu(\mathrm{d}u) q_r(D(z-u)) q_{t-r}(D(z_1-z)) q_{s'-r}(D(y-z)) q_{s-s'}(D(x-y)) \\
&\quad q_{t-s}(D(x-z_2)) q_{t'-s}(D(x-z_3)) q_{t'-s'}(D(y-z_4)) f_\varepsilon(z_1-z_2) f_\varepsilon(z_3-z_4) \\
&\quad (\det(D))^7\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\bar{z}\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}s'\, \mathrm{d}t\, \mathrm{d}t' |\mu| \int q_r(z) q_{t-r}(z_1-z) q_{s'-r}(y-z) \\
&\quad q_{s-s'}(x-y) q_{t-s}(x-z_2) q_{t'-s}(x-z_3) q_{t'-s'}(y-z_4) f_\varepsilon^D(z_1-z_2) \\
&\quad f_\varepsilon^D(z_3-z_4)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\bar{z}\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}s'\, \mathrm{d}t\, \mathrm{d}t',
\end{aligned} \tag{8.4}$$

where q denotes the Brownian transition density,

$$D = \sqrt{A^{-1}(z)}, \quad A(z) = \{a_{ij}(z)\}$$

and

$$f^D(x) = f(D^{-1}x)$$

Comparing Eq. (8.4) to the calculations in Eq. (3.11) we see that our last integral is

$$\frac{1}{64\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right)\left(\int f^D(y)\,\mathrm{d}y\right)\left(\int f^D(z)\,\mathrm{d}z\right)+\mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$
$$=\frac{1}{64\pi^4}\lg^2\left(\frac{1}{\varepsilon}\right)(\det(D))^2+0\lg\left(\frac{1}{\varepsilon}\right), \tag{8.5}$$

which proves Eq. (8.2); hence Theorem 3. □

9. Superstable processes: Theorem 4

Let y_t denote the symmetric stable process in $\mathbb{R}^d$ of index β with transition density

$$p_t^{(\beta)}(y)=\frac{1}{(2\pi)^d}\int \mathrm{e}^{\mathrm{i}py}\mathrm{e}^{-tp^\beta}\,\mathrm{d}p$$

so that

$$p_1^{(\beta)}(o)=\frac{1}{(2\pi)^d}\int \mathrm{e}^{-p^\beta}\,\mathrm{d}p=\frac{1}{\beta 2^{d-1}\pi^{d/2}}\frac{\Gamma(d/\beta)}{\Gamma(d/2)}.$$

With

$$G_\lambda^{(\beta)}(y)=\int_0^\infty \mathrm{e}^{-\lambda t}\,p_t(y)\,\mathrm{d}t,$$

we have

$$G_0^{(\beta)}(y)=\frac{\Gamma((d-\beta)/2)}{\Gamma(\beta/2)}\frac{1}{2^\beta\pi^{d/2}}\frac{1}{|y|^{d-\beta}}. \tag{9.1}$$

Our normalization has the property that

$$G_0^{(\alpha)}*G_0^{(\beta)}=G_0^{(\alpha+\beta)}. \tag{9.2}$$

Notice that with our normalization

$$p_t^{(2)}(y)=p_{2t}(y),$$

where p_t is the Brownian transition density, hence the Brownian Green's function is twice $G_0^{(2)}$.

When β is fixed, we often suppress it and set $G_1 \doteq G_1^{(\beta)}$. Theorem 4 will follow from Theorem 5 in the same manner that Theorem 1 followed from Theorem 2.

Theorem 5. *Let X_t be the superprocess over x_t, the symmetric stable process in R^d of index β killed at an independent exponential time.*
1. *If $d/2<\beta\leqslant d$ then as $\varepsilon\to 0$*

$$\int_0^\infty \langle f_\varepsilon(x-y),X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s-\zeta(\varepsilon)\int_0^\infty |X_s|\,\mathrm{d}s$$

converges in L^2, *where*

$$\zeta(\varepsilon)=\beta p_1(0)\lg\left(\frac{1}{\varepsilon}\right) \quad \textit{if} \ \beta=d, \\ \zeta(\varepsilon)=\frac{1}{\varepsilon^{d-\beta}}\int G_0^{(2\beta)}(x)f(x)\,\mathrm{d}x \quad \textit{if} \ \beta<d. \tag{9.3}$$

2. *If* $\beta=d/2$, *then if*

$$\gamma_\varepsilon=\int_0^\infty \langle f_\varepsilon(x-y), X_s(\mathrm{d}x)X_s(\mathrm{d}y)\rangle\,\mathrm{d}s-\alpha(\varepsilon)\int_0^\infty |X_s|\,\mathrm{d}s, \tag{9.4}$$

where

$$\alpha(\varepsilon)=\frac{1}{\varepsilon^\beta}\int G_0^{(\beta)}(x)f(x)\,\mathrm{d}x-\beta p_1(0)\lg\left(\frac{1}{\varepsilon}\right).$$

Then as $\varepsilon\to 0$, $\gamma_\varepsilon/\lg(1/\varepsilon)$ *converges in distribution and*

$$E_\mu(\mathrm{e}^{-\lambda\gamma_\varepsilon/\lg(1/\varepsilon)})\to \mathrm{e}^{|\mu|(1/2)(1-\sqrt{1-\wedge(d)\lambda^2})}$$

for λ *small*, *where*

$$\wedge(d)=\frac{2^{8-2d}}{\pi^d}\,\frac{1}{\Gamma^2(d/2)} \tag{9.5}$$

Theorem 5 will follow as in the proof of theorem 2 once we have computed the asymptotics of G and $G*G$, see the appendix of Rosen (1990).

We first use the resolvent equation to find

$$G=G_0-G_0*G. \tag{9.6}$$

If $d/2<\beta<d$, then it is easy to see that the last two terms are continuous, and then Eq. (9.2) gives that half of Eq. (9.3) referring to $\beta<d$.

If $\beta=d/2$, we use $G=G_0-G*G+G*G*G_0$ and proceed as in Eq. (2.11):

$$G*G(x)=\int_0^\infty \mathrm{e}^{-t}tp_t(x)\,\mathrm{d}t \\ =\int_0^1 \mathrm{e}^{-t}tp_t(x)\,\mathrm{d}t+\int_1^\infty \mathrm{e}^{-t}tp_t(x)\,\mathrm{d}t \tag{9.7}$$

and

$$\int_0^1 \mathrm{e}^{-t}tp_t(x)\,\mathrm{d}t=\int_0^1 tp_t(x)\,\mathrm{d}t+\int_0^1(\mathrm{e}^{-t}-1)tp_t(x)\,\mathrm{d}t. \tag{9.8}$$

Then, using scaling

$$\int_0^1 tp_t(x)\,\mathrm{d}t=\frac{1}{x^d}\int_0^1 tp_{t/x^\beta}(1)\,\mathrm{d}t \\ =\int_0^{1/x^\beta} tp_t(1)\,\mathrm{d}t$$

$$= \int_1^{1/x^\beta} tp_t(1)\,\mathrm{d}t + \int_0^1 tp_t(1)\,\mathrm{d}t$$
$$= \int_1^{1/x^\beta} tp_t(0)\,\mathrm{d}t + \int_1^{1/x^\beta} t(p_t(1) - p_t(0))\,\mathrm{d}t + \int_0^1 tp_t(1)\,\mathrm{d}t \tag{9.9}$$

and finally

$$\int_1^{1/x^\beta} tp_t(0)\,\mathrm{d}t = p_1(0)\int_1^{1/x^\beta} \frac{1}{t}\,\mathrm{d}t = \beta p_1(0)\lg\left(\frac{1}{|x|}\right) \tag{9.10}$$

Thus, from Eqs. (9.7)–(9.10) we find that for $\beta = d/2$,

$$G * G(x) = \beta p_1(0)\lg\left(\frac{1}{|x|}\right)$$

+ terms continuous in x which leads to $\alpha(\varepsilon)$.

Finally, we need the analogue of Eq. (3.11) when $\beta = d/2$:

$$K(\varepsilon) \sim \frac{1}{2^3}\int G(x)G * f_\varepsilon(x)G * G * f_\varepsilon(x)$$
$$= \frac{1}{2^3}\int_{2\varepsilon \leqslant |x| \leqslant 1/2} G_0 G_0 * f_\varepsilon G * G * f_\varepsilon(x)\,\mathrm{d}x + 0\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$
$$= \frac{1}{2^3}C^2(\beta)\beta p_1(0)\int_{2\varepsilon \leqslant |x| \leqslant 1/2} \frac{1}{x^{d-\beta}}\,\frac{1}{|x-y|^{d-\beta}}\lg\left(\frac{1}{|x-z|}\right)$$
$$f_\varepsilon(y)f_\varepsilon(z) + \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right)$$
$$= \frac{1}{2^3}C^2(\beta)\beta p_1(0)\frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{1}{2}\lg^2\left(\frac{1}{\varepsilon}\right) + \mathrm{O}\left(\lg\left(\frac{1}{\varepsilon}\right)\right), \tag{9.11}$$

where $C(\beta)$ is the coefficient of $1/X^{d-\beta}$ in Eq. (9.1). Putting all this together and using $d = 2\beta$, we find that

$$\wedge(d) = 2\cdot 2^3 C^2(\beta)\beta p_1(0)\frac{2\pi^{d/2}}{\Gamma(d/2)}\cdot\frac{1}{2}$$
$$= 2^3\frac{1}{(2\pi)^{2\beta}}\,\frac{1}{\beta 2^{d-1}\pi^{d/2}}\,\frac{1}{\Gamma(d/2)}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}$$
$$= \frac{2^{5-2d}}{\pi^d}\,\frac{1}{\Gamma^2(d/2)}. \tag{9.12}$$

We note that for $\beta = 2, d = 4$ this gives

$$\wedge(d) = 2^{5-2d}\frac{1}{\pi^4} = \frac{1}{2^3\pi^4},$$

consistent with our normalization as described at the beginning of this section, and the $2/\pi^4$ which appears in Theorem 2. □

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