ANNALES DE L'I. H. P., SECTION B

ENDRE CSÁKI
PÁL RÉVÉSZ
JAY ROSEN

# Functional laws of the iterated logarithm for local times of recurrent random walks on $Z^2$

*Annales de l'I. H. P., section B*, tome 34, n° 4 (1998), p. 545-563.
<http://www.numdam.org/item?id=AIHPB_1998__34_4_545_0>

*Ann. Inst. Henri Poincaré,*

Vol. 34, n° 4, 1998, p. 545-563. *Probabilités et Statistiques*

# Functional laws of the iterated logarithm for local times of recurrent random walks on $Z^2$

by

**Endre CSÁKI** *

Mathematical Institute of the Hungarian Academy of Sciences,
P.O.B. 127, Budapest, Hungary H-1364
csaki@novell.math-inst.hu

**Pál RÉVÉSZ**

Institut für Statistik und Wahrscheinlichkeitstheorie,
Technische Universität, Wien A-1040 Wien, Austria
revesz@ci.tuwien.ac.at

and

**Jay ROSEN** †

Department of Mathematics, College of Staten Island, CUNY,
Staten Island, NY 10314
jrosen3@mail.idt.net

---

ABSTRACT. – We prove functional laws of the iterated logarithm for $L_n^0$, the number of returns to the origin, up to step $n$, of recurrent random walks on $Z^2$ with slowly varying partial Green's function. We find two distinct functional laws of the iterated logarithm depending on the scaling used. In the special case of finite variance random walks, we obtain one limit set for $L_{n^x}^0/(\log n \log_3 n)\,;\ 0 \leq x \leq 1$, and a different limit set for $L_{xn}^0/(\log n \log_3 n)\,;\ 0 \leq x \leq 1$. In both cases the limit sets are classes of distribution functions, with convergence in the weak topology. © Elsevier, Paris

* This research was supported by Hungarian National Foundation for Scientific Research, Grants No. T 16384 and T 19346.

† This research was supported, in part, by grants from the National Science Foundation and PSC-CUNY.

*AMS Classifications:* 60 J 15, 60 F 17

RÉSUMÉ. – Nous démontrons des lois fonctionnelles du logarithme itéré pour $L_n^0$, le nombre de retours à l'origine avant l'instant $n$ d'une marche aléatoire récurrente sur $Z^2$ avec une fonction de Green à variation lente. Nous obtenons deux lois fonctionnelles différentes selon le changement d'échelle utilisé. Dans le cas particulier des marches aléatoires à variance finie, nous obtenons un ensemble limite pour $L_{n^x}^0/(\log n \log_3 n)$ ; $0 \le x \le 1$, et un ensemble limite différent pour $L_{xn}^0/(\log n \log_3 n)$ ; $0 \le x \le 1$. Dans les deux cas les ensembles limites sont des classes de fonctions de distribution, avec convergence pour la topologie faible. 

## 1. INTRODUCTION

Let $X_n$ be a symmetric adapted recurrent random walk on $Z^2$. We use $p_n(x)$ to denote the transition density of $X_n$, and let

$$g(n) = \sum_{k=0}^{n} p_k(0) \tag{1.1}$$

denote the partial Green's function. We can extend $g(t)$ to be a continuous monotone increasing function of $t \ge 0$. Recurrence means that $\lim_{t\to\infty} g(t) = \infty$. It is known, see e.g. Proposition 2.14 of [7], that

$$p_n(0) \le \frac{C}{n} \tag{1.2}$$

so that $g(n)$ is sub-logarithmic, i.e. $g(n) \le C \log n$. Throughout this paper we make the assumption that $g(n)$ is slowly varying at $\infty$. This will be satisfied in particular if $X_n$ is in the domain of attraction of a non-degenerate $R^2$-valued normal random variable.

As usual, $L_n^x$ will denote the local time of $X$ at $x$, i.e. the number of times $k \le n$ such that $X_k = x$. We extend $L_t^x$ to non-integer $t$ by linear interpolation. The following law of the iterated logarithm for the local time $L_n^0$ was proven in [7]:

$$\limsup_{n\to\infty} \frac{L_n^0}{g(n) \log_2 g(n)} = 1, \quad a.s. \tag{1.3}$$

where $\log_j$ denotes the j'th iterated logarithm. For the simple random walk in $Z^2$, (1.3) was proven by Erdős and Taylor [4]. See also Bertoin and

Caballero [1] for an alternate proof and generalization of (1.3). The object of this paper is to prove functional laws of the iterated logarithm for the local time $L_n^0$.

Let $\mathcal{M}$ be the set of functions $m(x)$, $0 \leq x \leq 1$, which are non-decreasing, right-continuous on $[0, 1)$ and left-continuous at $x = 1$. Let $\mathcal{M}^* \subseteq \mathcal{M}$ be the set of functions $m(x)$ in $\mathcal{M}$ such that $m(0) = 0$ and

$$\int_0^1 \frac{1}{x}\, dm(x) \leq 1. \tag{1.4}$$

We will always consider $\mathcal{M}$ with the weak topology, which is induced by the Lévy metric

$$d(m, \widetilde{m}) = \inf \{\epsilon > 0 \,|\, m(x - \epsilon) - \epsilon \leq \widetilde{m}(x) \leq m(x + \epsilon) + \epsilon\,,\, \forall\, x\},$$

where $m(x) = m(0)$ for $x < 0$ and $m(x) = m(1)$ for $x > 1$.

Let $\{t(n, x)\,;\, 0 \leq x \leq 1\}$, for $n = 1, 2, \ldots$, be any sequence of functions in $\mathcal{M}$ such that

$$\lim_{n \to \infty} \frac{g(t(n, x))}{g(n)} = x \tag{1.5}$$

for each $0 \leq x \leq 1$. Thus, for example, if $g(n) \sim c \log n$, we can take $t(n, x) = n^x$. If $g(n) \sim c\sqrt{\log n}$, we can take $t(n, x) = n^{x^2}$, while if $g(n) \sim c \log_2 n$, we can take $t(n, x) = e^{(\log n)^x}$. Our main theorem is

THEOREM 1. – *If*

$$f_n(x) = \frac{L^0_{t(n,x)}}{g(n) \log_2 g(n)}\,; \quad 0 \leq x \leq 1, \tag{1.6}$$

*then a.s. the set of limit points of* $\{f_n(x);\, n = 1, 2, \ldots\} \subseteq \mathcal{M}$ *is* $\mathcal{M}^*$.

The meaning of this statement is that there exists an event $\Omega_0 \subset \Omega$ of probability zero with the following two properties:

*Property* 1. – For any $\omega \notin \Omega_0$ and any sequence of integers $1 < \nu(1) < \nu(2) < \ldots$ there exist a random subsequence $\nu(k_j)$ and function $m \in \mathcal{M}^*$ such that

$$d(f_{\nu(k_j)},\ m) \to 0 \qquad (j \to \infty).$$

*Property* 2. – For any $m \in \mathcal{M}^*$ and $\omega \notin \Omega_0$ there exists a sequence of integers $\nu(k) = \nu(k, \omega, m)$ such that

$$d(f_{\nu(k)},\ m) \to 0 \qquad (k \to \infty).$$

We will prove Theorem 1 by showing that each of the above two properties holds. This is done in sections 2-3.

The special case of finite variance random walks on $Z^2$ deserves explicit mention:

COROLLARY 1. – *Let $X_n$ be a symmetric random walk on $Z^2$ with finite variance and let $|Q|$ denote the determinant of the covariance matrix $Q$ of $X_1$. If*

$$f_n(x) = \frac{L^0_{n^x}}{\log n \log_3 n}; \quad 0 \leq x \leq 1, \tag{1.7}$$

*then a.s. the set of limit points of $\{f_n(x); n = 1, 2, \ldots\} \subseteq \mathcal{M}$ is $(2|Q|^{1/2}/\pi)\mathcal{M}^*$.*

We want to contrast Theorem 1 with the functional law of the iterated logarithm for the local times $L^0_n$ of a symmetric random walk on $Z$ in the domain of attraction of a stable random variable of index $1 < \beta \leq 2$. Theorem 1.4 of [8], see also Theorem 1.4 of [7], says that a.s. the set of limit points of

$$f_n(x) = \frac{L^0_{xn}}{c(\beta) g(n/\log_2 g(n)) \log_2 g(n)}; \quad 0 \leq x \leq 1 \tag{1.8}$$

in the uniform topology is $\mathcal{M}^\beta$, where $c(\beta)$ is a universal constant and $\mathcal{M}^\beta \subseteq \mathcal{M}$ is the set of functions $f \in \mathcal{M}$ which are absolutely continuous with respect to Lebesgue measure, with

$$\int_0^1 |f'(x)|^\beta \, dx \leq 1.$$

Thus, $\mathcal{M}^2$ is the set of monotone functions in the usual Strassen class.

We note that for the random walks considered in Theorem 1 we have

$$g(n/\log_2 g(n)) \sim g(n), \tag{1.9}$$

see the proof of Theorem 1.1 in [7]. In comparing (1.6) with (1.8) we see that the scaling of $L^0_n$ in $n$, the topology and the limit sets are all quite different. If we use the scaling $L^0_{xn}$ in the case of recurrent random walks on $Z^2$ we obtain another limit set which we now describe.

Let

$$h(m_1, m_2, \tau; x) = \begin{cases} m_1 & \text{if} \quad 0 \leq x < \tau, \\ m_2 & \text{if} \quad \tau \leq x \leq 1. \end{cases} \tag{1.10}$$

Define the set of functions $\mathcal{M}^{\triangle} \subseteq \mathcal{M}$ as follows:

$$\{m(x),\ 0 \leq x \leq 1\} \in \mathcal{M}^{\triangle}$$

if and only if

$$m(x) = h(m_1, m_2, \tau; x)$$

for a triple $0 \leq m_1 \leq m_2 \leq 1,\ 0 < \tau \leq 1$.

We have the following result.

THEOREM 2. – *Let*

$$\widetilde{f}_n(x) = \frac{L^0_{xn}}{g(n) \log_2 g(n)}; \qquad 0 \leq x \leq 1.$$

*Then the set of limit points of the sequence* $\{\widetilde{f}_n(x),\ n = 1, 2, \ldots\} \subseteq \mathcal{M}$ *is* $\mathcal{M}^{\triangle}$.

The proof of Theorem 2 is given in sections 4-6.

We note that similar functional laws of the iterated logarithm hold for the local times of 1-dimensional symmetric random walks and Lévy processses in the domain of attraction of a Cauchy random variable. In these cases $g(n)$ need not satisfy (1.9). Rather than state a general theorem we only mention that under the conditions of Theorem 1.3 of [7] the proofs of this paper can be used to show that if

$$\bar{f}_n(x) = \frac{L^0_{t(n,x)}}{g(n/\log_2 g(n)) \log_2 g(n)}; \quad 0 \leq x \leq 1 \tag{1.11}$$

then a.s. the set of limit points of $\{\bar{f}_n(x);\ n = 1, 2, \ldots\} \subseteq \mathcal{M}$ is $\mathcal{M}^*$.

## 2. THEOREM 1: PROOF OF PROPERTY 1

We begin by recalling certain results from [7]. In the following, $\delta$ will denote an arbitrarily small positive number, whose value may change from line to line. By Lemma 2.5 of [7], for any $\delta > 0$ and $x \geq x_0(\delta)$ sufficiently large we have

$$P\left\{\frac{L^0_t}{g(t)} \geq x\right\} \leq e^{-(1-\delta)x}, \tag{2.1}$$

while by Lemma 2.7 of [7], for $x \geq x_0(\delta)$ sufficiently large (but much smaller than $t$, see the exact statement in [7] for the details, which won't present a problem here)

$$P\left\{\frac{L_t^0}{g(t)} \geq x\right\} \geq e^{-(1+\delta)x}. \tag{2.2}$$

Combining these we see that for any $b > 0$ and $y > x > 0$

(2.3)
$$\begin{aligned}
&P\left\{L^0_{t(n,x)} < bg(n)\log_2 g(n) \leq L^0_{t(n,y)}\right\}\\
&= P\left\{L^0_{t(n,y)} \geq bg(n)\log_2 g(n)\right\} - P\left\{L^0_{t(n,x)} \geq bg(n)\log_2 g(n)\right\}\\
&= P\left\{\frac{L^0_{t(n,y)}}{yg(n)} \geq (b/y)\log_2 g(n)\right\} - P\left\{\frac{L^0_{t(n,x)}}{xg(n)} \geq (b/x)\log_2 g(n)\right\}\\
&\geq \left(e^{-(1+\delta)(b/y)\log_2 g(n)} - e^{-(1-\delta)(b/x)\log_2 g(n)}\right)\\
&\geq e^{-(1+\delta)(b/y)\log_2 g(n)}\left(1 - e^{-b((1-\delta)/x-(1+\delta)/y)\log_2 g(n)}\right)\\
&\geq (1/2)e^{-(1+\delta)(b/y)\log_2 g(n)}
\end{aligned}$$

for $n$ sufficiently large, depending on $b, x, y$ and $\delta > 0$. Furthermore, Lemma 2.7 of [7] also provides the following lower bound for walks starting from $z$

$$P^z\left\{\frac{L_t^0}{g(t)} \geq x\right\} \geq (a(z, 2t/(3x)) - e^{-\delta x})^+ e^{-(1+\delta)x}. \tag{2.4}$$

where
$$a(z,n) = \frac{\sum_{k=0}^n p_k(z)}{g(n)}$$
and $x^+ = \max(x, 0)$. We extend $a(z,t)$ to non-integer $t$ by linear interpolation.

Our proof of Theorem 1 will be based on the following:

LEMMA 1.

(i) *For* $0 < x_1 < x_2 < \ldots < x_r \leq 1$, $0 < v_1 \leq v_2 \leq \ldots \leq v_r$, *we have for $n$ large enough*

$$\begin{aligned}
&P\{L^0_{t(n,x_i)} > v_i\, g(n)\log_2 g(n),\ i = 1,\ldots,r\}\\
&\leq \exp\left\{\left(-\frac{v_1}{x_1} - \frac{v_2 - v_1}{x_2} - \cdots - \frac{v_r - v_{r-1}}{x_r} + \delta\right)\log_2 g(n)\right\}.
\end{aligned} \tag{2.5}$$

(ii) *For* $0 < x_1 < x_2 < \ldots < x_r \le 1$, $0 < v_1 < u_1 < v_2 < u_2 < \ldots < v_{r-1} < u_{r-1} < v_r < u_r$, *we have for* $n$ *large enough*

$$\begin{aligned}(2.6)\qquad & P^z\{v_i\, g(n)\log_2 g(n) < L^0_{(1-\delta)t(n,x_i)} \\ & \qquad \le L^0_{t(n,x_i)} \le u_i\, g(n)\log_2 g(n),\ i=1,\ldots,r\} \\ & \ge 2^{-r}\Big(a(z, x_1 n/(2v_1\log_2 g(n))) - e^{-\delta\log_2 g(n) v_1/x_1}\Big)^+ \\ & \exp\left\{\left(-\frac{v_1}{x_1} - \frac{v_2-v_1}{x_2} - \cdots - \frac{v_r - v_{r-1}}{x_r} - \delta\right)\log_2 g(n)\right\}.\end{aligned}$$

*Proof of Lemma* 1. – Let $q_0 = 0$, $q_i = [v_i\, g(n)\log_2 g(n)]$, $i = 1,\ldots,r$, where $[x]$ denotes the integer part of $x$, and let $\rho_0 = 0 < \rho_1 < \rho_2 < \ldots$ be the times of returns to the origin of our random walk. Then

$$\begin{aligned}&(2.7)\\ &P\{L^0_{t(n,x_i)} > v_i\, g(n)\log_2 g(n),\ i=1,\ldots,r\} \\ &\le P\{\rho_{q_i} \le t(n,x_i),\ i=1,\ldots,r\} \\ &\le P\{\rho_{q_i} - \rho_{q_{i-1}} \le t(n,x_i),\ i=1,\ldots,r\} \\ &= P\{\rho_{q_1} \le t(n,x_1)\}P\{\rho_{q_2-q_1} \le t(n,x_2)\}\cdots P\{\rho_{q_r-q_{r-1}} \le t(n,x_r)\} \\ &= P\{L^0_{t(n,x_1)} \ge q_1\}P\{L^0_{t(n,x_2)} \ge q_2 - q_1\}\cdots P\{L^0_{t(n,x_r)} \ge q_r - q_{r-1}\}.\end{aligned}$$

Now (2.5) follows by applying (2.1).

To show (2.6), set $t_0 = u_0 = 0$ and let $s_i = [v_i\, g(n)\log_2 g(n)] + 1$, $t_i = [u_i\, g(n)\log_2 g(n)]$, $i = 1,\ldots,r$. Then $t_0 = 0 < s_1 < t_1 < s_2 < t_2 < \ldots < s_{r-1} < t_{r-1} < s_r < t_r$ for large enough $n$. Using $z_1 = z$ and $z_j = 0$ for all $j \ne 1$, and the conventions $t(n,x_0) = 0$, $x_{r+1} = t(n,x_{r+1}) = t(n,(1-\delta)x_{r+1}) = \infty$ we have

$$\begin{aligned}(2.8)\qquad & P^z\{v_i\, g(n)\log_2 g(n) < L^0_{(1-\delta)t(n,x_i)} \\ & \qquad \le L^0_{t(n,x_i)} \le u_i\, g(n)\log_2 g(n),\ i=1,\ldots,r\} \\ & \ge P^z\{s_i < L^0_{(1-\delta)t(n,x_i)} \le L^0_{t(n,x_i)} \le t_i,\ i=1,\ldots,r\} \\ & \ge P^z\{\rho_{s_1} < (1-\delta)t(n,x_1) < t(n,x_1) \le \rho_{t_1} < \rho_{s_2} < \ldots \\ & \qquad < (1-\delta)t(n,x_{r-1}) < t(n,x_{r-1}) \le \rho_{t_{r-1}} < \rho_{s_r} \\ & \qquad < (1-\delta)t(n,x_r) < t(n,x_r) \le \rho_{t_r}\} \\ & \ge P^z\Big\{\rho_{s_i} - \rho_{t_{i-1}} < (1/2-\delta)t(n,x_i), \\ & \qquad t(n,x_i) - t(n,x_{i-1}) \le \rho_{t_i} - \rho_{s_i} \\ & \qquad < \frac{t(n,x_{i+1})}{2} - (1-\delta)t(n,x_i),\ i=1,\ldots,r\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{r} P^{z_i}\{\rho_{s_i - t_{i-1}} < (1/2 - \delta) t(n, x_i)\} \\
&\quad P\Big\{ t(n, x_i) - t(n, x_{i-1}) \\
&\qquad \le \rho_{t_i - s_i} < \frac{t(n, x_{i+1})}{2} - (1 - \delta) t(n, x_i) \Big\} \\
&\ge \prod_{i=1}^{r} P^{z_i}\{\rho_{s_i - t_{i-1}} < t(n, (1 - \delta) x_i)\} \\
&\quad \times P\{t(n, x_i) < \rho_{t_i - s_i} \le t(n, (1 - \delta) x_{i+1})\} \\
&\ge \prod_{i=1}^{r} P^{z_i}\Big\{ L^0_{t(n,(1-\delta)x_i)} > s_i - t_{i-1} \Big\} \\
&\quad \times P\Big\{ L^0_{t(n,x_i)} < t_i - s_i \le L^0_{t(n,(1-\delta)x_{i+1})} \Big\} \\
&\ge 2^{-r} \Big( a(z, x_1 n / (2 v_1 \log_2 g(n))) - e^{-\delta \log_2 g(n) v_1 / x_1} \Big)^+ \\
&\quad \prod_{i=1}^{r} e^{-(1+\delta) \log_2 g(n)(v_i - u_{i-1})/x_i} e^{-(1+\delta) \log_2 g(n)(u_i - v_i)/x_{i+1}} \\
&\ge 2^{-r} \Big( a(z, x_1 n / (2 v_1 \log_2 g(n))) - e^{-\delta \log_2 g(n) v_1 / x_1} \Big)^+ \\
&\quad \times e^{-(1+\delta) \log_2 g(n) v_1 / x_1} \\
&\quad \prod_{i=2}^{r} e^{-(1+\delta) \log_2 g(n)(v_i - u_{i-1})/x_i} e^{-(1+\delta) \log_2 g(n)(u_{i-1} - v_{i-1})/x_i} \\
&\ge 2^{-r} \Big( a(z, x_1 n / (2 v_1 \log_2 g(n))) - e^{-\delta \log_2 g(n) v_1 / x_1} \Big)^+ \\
&\quad \times e^{-(1+\delta) \log_2 g(n) v_1 / x_1} \\
&\quad \prod_{i=2}^{r} e^{-(1+\delta) \log_2 g(n)(v_i - v_{i-1})/x_i}
\end{aligned}
$$

where we have applied (2.2), (2.3) and (2.4). This completes the proof of Lemma 1.

Now we prove Property 1.

Let $0 < x_1 < x_2 < \ldots < x_r \le 1$, $0 < z_1 < z_2 < \ldots < z_r$ such that

(2.9)
$$b = \frac{z_1}{x_1} + \frac{z_2 - z_1}{x_2} + \cdots + \frac{z_r - z_{r-1}}{x_r} > 1.$$

Define $n_k$ by $g(n_k) = d^k$ with $1 < d < b$ and

(2.10)
$$A_k = \{L^0_{t(n_{k+1}, x_i)} > z_i\, g(n_k) \log_2 g(n_k),\ i = 1, \ldots, r\}.$$

It follows from (2.5) that

$$(2.11)\quad P\{A_k\} \le \exp\left\{\frac{1}{d}\left(-\frac{z_1}{x_1}-\frac{z_2-z_1}{x_2}-\cdots-\frac{z_r-z_{r-1}}{x_r}+\delta\right)\log_2 g(n_k)\right\}. \le ck^{-(b-\delta)/d}.$$

Since $b/d > 1$ this shows that $P\{A_k \text{ i.o.}\} = 0$. By interpolating for $n_k \le n < n_{k+1}$, this implies

$$P\{f_n(x_i) > z_i,\ i = 1,\ldots,r \text{ i.o.}\} = 0. \tag{2.12}$$

By choosing $x_i,\ z_i$ rational, one can see that there is a universal set $\Omega_1$ of probability 1 such that for all $\omega \in \Omega_1$ and all rational $x_i,\ z_i$ we have that

$$\{f_n(x_i) > z_i,\ i = 1,\ldots,r\} \tag{2.13}$$

occur only finitely often.

Since $\{f_n\}$ is a sequence of a.s. bounded functions in $\mathcal{M}$, by the Helly-Bray theorem every subsequence has a further subsequence which is convergent in the Lévy metric. Its limit $\hat{m}$ is in $\mathcal{M}$. Suppose that for an $\omega \in \Omega$, $\hat{m} \notin \mathcal{M}^*$, i.e. $\int_0^1 \frac{1}{x}\,d\hat{m}(x) > 1$. Then one can find rational $x_i,\ z_i,\ i = 1,\ldots,r$ such that $z_i \le \hat{m}(x_i),\ i = 1,\ldots,r$ and at least for some $i$, $z_i < \hat{m}(x_i)$ and moreover,

$$\frac{z_1}{x_1}+\frac{z_2-z_1}{x_2}+\cdots+\frac{z_r-z_{r-1}}{x_r} > 1. \tag{2.14}$$

In view of (2.12) and since $\hat{m}$ is supposed to be a limit point of $\{f_n\}$, we conclude that $\omega \notin \Omega_1$. This proves Property 1.

## 3. THEOREM 1: PROOF OF PROPERTY 2

Now we turn to the proof of Property 2.

Let $m^* \in \mathcal{M}^*$ such that $\int_0^1 \frac{1}{x}\,dm^*(x) < 1$. Given small $\varepsilon > 0$, we can find $0 < x_1 < \ldots < x_r \le 1$ and $0 < z_1 < \ldots < z_r$ such that

$$z_i + \varepsilon < z_{i+1} - \varepsilon,\ i = 1,\ldots,r-1 \tag{3.1}$$

and

$$\frac{z_1}{x_1}+\frac{z_2-z_1}{x_2}+\cdots+\frac{z_r-z_{r-1}}{x_r} < 1 \tag{3.2}$$

and for any $f \in \mathcal{M}^*$

$$\{z_i - \varepsilon < f(x_i) < z_i + \varepsilon,\ i = 1, \ldots, r\} \subset \{d(f, m^*) < 3\varepsilon\}. \tag{3.3}$$

LEMMA 2. – *Define the events* $A_n^*$ *by*

$$A_n^* = \{z_i - \varepsilon < f_n(x_i) < z_i + \varepsilon,\ i = 1, \ldots, r\} \tag{3.4}$$

*where* $\varepsilon > 0,\ 0 < x_1, < \ldots < x_r \leq 1,\ z_i,\ x_i$ *satisfy (3.1) and (3.2). Then*

$$P\{A_n^*\ i.o.\} = 1. \tag{3.5}$$

*Proof of Lemma* 2. – We follow the proof of Theorem 1.1 of [7], to which the reader can refer for further details. Define $n_k$ by $g(n_k) = \theta^k$ with $1 < \theta$. As usual, we let $\mathcal{F}_n$ denote the $\sigma$-algebra generated by $X_1, \ldots, X_n$. By the extended Borel-Cantelli Lemma (see Corollary 5.29 in [2])

$$\{A_{n_k}^*\ i.o.\} = \left\{\sum_{k=1}^{\infty} P\{A_{n_k}^* \,|\, \mathcal{F}_{n_{k-1}}\} = \infty\right\}, \tag{3.6}$$

so that to prove our Lemma it suffices to show that for sufficiently large $\theta$

$$P\left\{\sum_{k=1}^{\infty} P\{A_{n_k}^* \,|\, \mathcal{F}_{n_{k-1}}\} = \infty\right\} \geq 1 - 3/\theta \tag{3.7}$$

and using (1.3) we see that it suffices to show that

$$P\left\{\sum_{k=1}^{\infty} P\{A_{n_k}^\sharp \,|\, \mathcal{F}_{n_{k-1}}\} = \infty\right\} \geq 1 - 3/\theta \tag{3.8}$$

for all $\theta$ sufficiently large, where

$$A_{n_k}^\sharp = \left\{z_i - \varepsilon < \frac{L_{t(n_k, x_i)}^0 - L_{n_{k-1}}^0}{g(n_k) \log_2 g(n_k)} < z_i + \varepsilon,\ i = 1, \ldots, r\right\}. \tag{3.9}$$

Using the Markov property it suffices to show that the event

(3.10)

$$\sum_{k=1}^{\infty} P^{X_{n_{k-1}}}\left\{z_i - \varepsilon < \frac{L_{t(n_k, x_i) - n_{k-1}}^0}{g(n_k) \log_2 g(n_k)} < z_i + \varepsilon,\ i = 1, \ldots, r\right\} = \infty$$

has probability $\geq 1 - 3/\theta$. We note that $\lim_{k\to\infty} t(n_k, x_1)/n_{k-1} = \infty$, as follows from the proof of Proposition 2.9 in [7]. Hence

$$(1-\delta)t(n_k, x_i) < t(n_k, x_i) - n_{k-1} < t(n_k, x_i).$$

Thus the left hand side of (3.10) is no less than

$$\sum_{k=1}^{\infty} P^{X_{n_{k-1}}} \Big\{ (z_i - \varepsilon)\, g(n_k) \log_2 g(n_k) < L^0_{(1-\delta)t(n_k, x_i)} \leq L^0_{t(n_k, x_i)} \\ \leq (z_i + \varepsilon)\, g(n_k) \log_2 g(n_k),\ i = 1, \ldots, r \Big\}.$$

By (2.6) this can be bounded from below by

$$C \sum_{k=1}^{\infty} \left( a(X_{n_{k-1}}, x_1 n_k/(2v_1 \log k)) - \frac{1}{k^{\delta'}} \right)^+ \frac{1}{k^\alpha} \tag{3.11}$$

where

$$\alpha = \frac{z_1 - \varepsilon}{x_1} + \frac{z_2 - z_1}{x_2} + \ldots + \frac{z_r - z_{r-1}}{x_r} + \delta < 1 \tag{3.12}$$

and $\delta' > 0$. It is easily checked using (1.9) that

$$E\big(a(X_{n_{k-1}}, x_1 n_k/(2v_1 \log k))\big) \\ = \frac{g(n_{k-1} + x_1 n_k/(2v_1 \log k)) - g(n_{k-1})}{g(x_1 n_k/(2v_1 \log k))} \geq 1 - 2/\theta$$

for $k$ and $\theta$ large, so that the expectation of (3.11) diverges. Since clearly $a(X_{n_{k-1}}, x_1 n_k/(2v_1 \log k)) \leq 1$ we can use the Paley-Zygmund inequality (see e.g. Inequality II, page 8 of [5]) to show that (3.10) has probability $\geq 1 - 3/\theta$. This concludes the proof of Lemma (2).

It follows from Lemma (2) and (3.3) that for $m^* \in \mathcal{M}^*$ we have $d(f_n, m^*) < \varepsilon$ i.o. with probability 1, but the exceptional set of probability 0 may depend on $m^*$. To show that this is not the case we note that one can choose a countable set of functions $m^*$ with $\int_0^1 \frac{1}{x}\, dm^*(x) < 1$, dense in $\mathcal{M}^*$ (with respect to the Lévy metric). The countable union of the exceptional sets of probability 0 is also of probability 0, and obviously this exceptional set is universal for all $m^* \in \mathcal{M}^*$. By choosing $\varepsilon = \varepsilon_k = 1/k$, for all $\omega$ not in this exceptional set one can find a sequence $\{\nu_k\}$ such that $d(f_{\nu_k}, m^*) < 1/k$, i.e. $f_{\nu_k} \to m^*$ as $k \to \infty$ in the Lévy metric, so $m^*$ is a limit point and we have Property 2. This completes the proof of Theorem 1.

## 4. ESTIMATES ON NON-RETURN

The next two sections develop material needed for the proof of Theorem 2.

The following Lemma and its proof are simple translations of [4] to our context. Let $\rho$ denote the length of the first excursion from the origin.

LEMMA 3.

$$P\{\rho > n\} = 1/g(n) + O(1/g^2(n)). \tag{4.1}$$

*Proof of Lemma* 3. – Let

$$\gamma(n) = P\{\rho > n\}.$$

Considering the last return to the origin we have

$$\sum_{k=0}^{n} \gamma(n-k) p_k(0) = 1 \tag{4.2}$$

so that

$$\gamma(n) \sum_{k=0}^{n} p_k(0) \leq 1 \tag{4.3}$$

which shows that

$$\gamma(n) \leq 1/g(n). \tag{4.4}$$

On the other hand, (4.2) also shows that

(4.5)

$$\gamma(n/2) \sum_{k=0}^{n/2} p_k(0) + \gamma(n/g(n)) \sum_{k=n/2}^{n-n/g(n)} p_k(0) + \sum_{k=n-n/g(n)}^{n} p_k(0) \geq 1.$$

Now

(4.6)

$$\sum_{k=n-n/g(n)}^{n} p_k(0) \leq C \sum_{k=n-n/g(n)}^{n} 1/k \leq C/n \sum_{k=n-n/g(n)}^{n} 1 = C/g(n)$$

and using the fact that

$$0 \leq g(n) - g(n/g(n)) \leq C \sum_{k=n/g(n)}^{n} \frac{1}{k} \leq C \log g(n) \tag{4.7}$$

so that

$$g(n) \leq Cg(n/g(n))$$

we have

$$\begin{aligned}(4.8) \qquad & \gamma(n/g(n)) \sum_{k=n/2}^{n-n/g(n)} p_k(0) \\ & \leq \gamma(n/g(n)) \sum_{k=n/2}^{n} p_k(0) \\ & \leq C/g(n/g(n)) \sum_{k=n/2}^{n} 1/k \\ & \leq C/g(n).\end{aligned}$$

Hence

$$\gamma(n/2)g(n/2) \geq 1 - C/g(n). \tag{4.9}$$

Our lemma then follows from the slow variation of $g$.

## 5. LARGE EXCURSIONS

Introduce the following notation:

$$\begin{aligned} \rho_0 &= 0, \\ \rho_1 &= \min\{n : \ n > 0, \ X_n = (0,0)\}, \\ \rho_2 &= \min\{n : \ n > \rho_1, \ X_n = (0,0)\}, \\ &\vdots \\ r(k) &= \rho_k - \rho_{k-1} \quad (k = 1, 2, \ldots). \end{aligned}$$

Let

$$M_n^{(1)} \geq M_n^{(2)} \geq \ldots \geq M_n^{(L_n^0+1)}$$

be the order statistics of the sequence

$$r(1), r(2), \ldots, r(L_n^0), n - \rho_{L_n^0}.$$

Now we have

LEMMA 4.

$$\lim_{n\to\infty} \frac{M_n^{(1)} + M_n^{(2)}}{n} = 1.$$

*Proof.* – Define $n_j$ by $g(n_j) = j$. Using the fact that as in (4.7)

$$0 \le j - g(n_j/j^2) = g(n_j) - g(n_j/j^2) \le C \sum_{k=n_j/j^2}^{n_j} \frac{1}{k} \le C \log j$$

together with Lemma 3 we have

$$\begin{aligned}
\phi(j) &\stackrel{def}{=} P\{n_j(g(n_j))^{-2} < \rho_1 \le n_{j+1}\} \\
&= P\left\{\frac{n_j}{j^2} < \rho_1 \le n_{j+1}\right\} \\
&= P\left\{\rho_1 > \frac{n_j}{j^2}\right\} - P\{\rho_1 > n_{j+1}\} \\
&\le (g(n_j/j^2))^{-1} - (j+1)^{-1} + Cj^{-2} \\
&\le C(\log j)j^{-2}
\end{aligned}$$

if $j$ is big enough. Let

$$\begin{aligned}
N(n) &= [g(n)\log g(n)], \\
\kappa(n) &= \#\{i :\ i \le N(n),\ n(g(n))^{-2} < r(i) \le n\}, \\
\kappa^*(n_j) &= \#\{i :\ i \le N(n_{j+1}),\ n_j(g(n_j))^{-2} < r(i) \le n_{j+1}\}.
\end{aligned}$$

Then, since $N(n_j) = j\log j$, we have

$$\begin{aligned}
P\{\kappa^*(n_j) > 1\} &= 1 - (1-\phi(j))^{N(n_{j+1})} - N(n_{j+1})(1-\phi(j))^{N(n_{j+1})-1}\phi(j) \\
&\sim \frac{(N(n_{j+1})\phi(j))^2}{2} \le C\frac{(\log j)^4}{j^2}.
\end{aligned}$$

Thus $\kappa^*(n_j) \le 1$ a.s. for all but finitely many $j$. Now take $n_j \le n \le n_{j+1}$. Since

$$N(n) \le N(n_{j+1})$$

and

$$n_j(g(n_j))^{-2} \le n(g(n))^{-2} < n \le n_{j+1}$$

we obtain that

$$\kappa(n) \le \kappa^*(n_j) \le 1.$$

Since $L_n^0 \leq N(n)$ (cf. (1.3)) there are no more than $N(n)$ excursions before time $n$. Thus the sum of those elements of the sequence

$$r(1), r(2), \ldots, r(L_n^0), n - \rho_{L_n^0} \tag{5.1}$$

which are no larger than $n(g(n))^{-2}$ is bounded by

$$N(n)n(g(n))^{-2} = (n \log g(n))/g(n) = o(n).$$

Hence the fact that $\kappa(n) \leq 1$ implies the Lemma.

This Lemma and its proof are essentially the same as the corresponding Lemma and proof in Révész and Willekens [9].

When $n = \rho_m$ for some $m$, the last element in (5.1) vanishes and the above proof immediately implies the following corollary.

COROLLARY 2.

$$\lim_{m \to \infty} \frac{M_{\rho_m}^{(1)}}{\rho_m} = 1 \qquad a.s.$$

COROLLARY 3. – *For any $\varepsilon > 0$ let $\lambda(n, \varepsilon)$ resp. $\mu(n, \varepsilon)$ be the last resp. first return of the random walk $X.$ to the origin before resp. after $(1+\varepsilon)\rho_n$. Then for any $K > 0$ there exists an $n_0 = n_0(K, \omega)$ such that*

$$\mu(n, \varepsilon) - \lambda(n, \varepsilon) \geq K\rho_n$$

*if $n \geq n_0$.*

*Proof of Corollary* 3. – Assume on the contrary that we can find an infinite sequence $n_j$ such that $\mu(n_j, \varepsilon) - \lambda(n_j, \varepsilon) < K\rho_{n_j}$ for some K. By Corollary 2 we can find an excursion in $(0, \rho_{n_j})$ with length $M_{n_j}^- \sim \rho_{n_j}$. Similarly, since $\mu(n_j, \varepsilon)$ is itself of the form $\rho_n$ for some $n$, we can find an excursion in $(\rho_{n_j}, \mu(n_j, \varepsilon))$ of length $M_{n_j}^+ \sim \mu(n_j, \varepsilon) - \rho_{n_j}$. Since our assumption says that $(1+\epsilon)\rho_{n_j} \leq \mu(n_j, \varepsilon) < (1+\epsilon+K)\rho_{n_j}$, so that

$$\frac{M_{n_j}^-}{\mu(n_j, \varepsilon)} \geq \frac{1}{1+\epsilon+K}, \quad \frac{M_{n_j}^+}{\mu(n_j, \varepsilon)} \geq \frac{\epsilon}{1+\epsilon+K},$$

the existence of the two (distinct) excursions of length $M_{n_j}^-, M_{n_j}^+$ would contradict Corollary 2 applied to $\rho_n = \mu(n_j, \varepsilon)$.

## 6. PROOF OF THEOREM 2

We will prove Theorem 2 by showing that there exists an event $\Omega_0 \subset \Omega$ of probability zero with the following two properties:

*Property* 1′. – For any $\omega \notin \Omega_0$ and any sequence of integers $1 < \nu(1) < \nu(2) < \ldots$ there exist a random subsequence $\nu(k_j)$ and function $m \in \mathcal{M}^{\triangle}$ such that

$$d(\widetilde{f}_{\nu(k_j)},\ m) \to 0 \qquad (j \to \infty).$$

*Property* 2′. – For any $m \in \mathcal{M}^{\triangle}$ and $\omega \notin \Omega_0$ there exists a sequence of integers $\nu(k) = \nu(k, \omega, m)$ such that

$$d(\widetilde{f}_{\nu(k)},\ m) \to 0 \qquad (k \to \infty).$$

By the Helly-Bray theorem, for any subsequence of the sequence $\{\widetilde{f}_n(x)\}$ there exists a further subsequence which is convergent in the Lévy metric. Lemma 4 clearly implies that the limit is in $\mathcal{M}^{\triangle}$ with probability 1. This proves Property 1′.

To show Property 2′ we prove that any

$$m(x) = h(m_1, m_2, \tau; x) \in \mathcal{M}^{\triangle}$$

is a limit point of the sequence $\{\widetilde{f}_n(x)\}$ with probability 1.

LEMMA 5. – *For almost all $\omega$, and any $0 < m_2 < 1$ and $\varepsilon > 0$ there exists a sequence of integers $n_1 = n_1(\omega, m_2, \varepsilon) < n_2 = n_2(\omega, m_2, \varepsilon) < \ldots$ such that*

$$m_2 - \varepsilon < \frac{L^0_{n_i}}{g(n_i)\log_2 g(n_i)} < m_2 + \varepsilon \qquad a.s.,$$

$$X_{n_i} = (0, 0) \qquad (i = 1, 2, \ldots).$$

*Proof of Lemma* 5. – Let $\varepsilon > 0$ be so small that $\varepsilon < m_2 - \varepsilon$ and $m_2 + \varepsilon < 1 - \varepsilon$. Clearly, there exist sequences $\{n_i^{(1)} = n_i^{(1)}(\omega, m_2, \varepsilon)\}$ and $\{n_i^{(2)} = n_i^{(2)}(\omega, m_2, \varepsilon)\}$ such that

$$\frac{L^0_{n_i^{(1)}}}{g(n_i^{(1)})\log_2 g(n_i^{(1)})} < \varepsilon$$

and

$$1 - \varepsilon < \frac{L^0_{n_i^{(2)}}}{g(n_i^{(2)})\log_2 g(n_i^{(2)})} < 1 + \varepsilon.$$

We may assume that the two sequences are alternating, i.e. $n_i^{(1)} < n_i^{(2)} < n_{i+1}^{(1)}, i = 1, 2, \ldots$. It is intuitively clear that we can find a subsequence with the desired property by interpolating between these two sequences. This argument can be made precise as follows. Define $n_i$ by

$$n_i = \min\{n : n > n_i^{(1)}, L_n^0 > (m_2 - \varepsilon)g(n)\log_2 g(n)\}.$$

Then obviously $n_i < n_i^{(2)}$ and $L_{n_i}^0 = L_{n_i-1}^0 + 1$. Hence $X_{n_i} = (0, 0)$, since $L_n^0$ increases at return points only. Moreover,

$$L_{n_i}^0 = L_{n_i-1}^0 + 1 \leq (m_2 - \varepsilon)g(n_i - 1)\log_2 g(n_i - 1) + 1 \leq$$

$$\leq m_2 g(n_i)\log_2 g(n_i),$$

i.e. we have the Lemma.

Next we prove

LEMMA 6. – *For any* $0 < \tau < 1$, $0 < m_1 < m_2 < 1$ *and* $\varepsilon > 0$ *we have*

$$P\{d(\widetilde{f}_n, h) < \varepsilon \quad \text{i.o.}\} = 1,$$

*where* $d$ *is the Lévy metric and* $h = h(m_1, m_2, \tau; x)$ *is defined by (1.10).*

*Proof of Lemma* 6. – Let

$$\kappa(N) = \min\{k : k \geq 1, \rho_k - \rho_{k-1} = \max_{1\leq j\leq N}(\rho_j - \rho_{j-1})\},$$

i.e. the (random) index of the longest of the first $N$ excursions and define the events

$$A_N = \left\{m_2 - \varepsilon < \frac{N}{g(\rho_N)\log_2 g(\rho_N)} < m_2 + \varepsilon\right\}, \tag{6.1}$$

$$B_N = \left\{\frac{m_1}{m_2} - \varepsilon < \frac{\kappa(N)}{N} < \frac{m_1}{m_2} + \varepsilon\right\}, \tag{6.2}$$

$$C_N = \{L_{\rho_N + \alpha_N} - L_{\rho_N} \leq \varepsilon N\}, \tag{6.3}$$

where $\alpha_N$ is defined by $g(\alpha_N) = 2N$.

We use the following

LEMMA 7. – *Let* $\{A_N\}_{N=1}^{\infty}$ *and* $\{B_N\}_{N=1}^{\infty}$ *be two sequences of events such that* $P\{A_N \text{ i.o.}\} > 0$ *and* $\liminf_{N\to\infty} P\{B_N\} > 0$. *Assume further that either one of the following two conditions hold:*

(i) $B_N$ *is independent of* $\{A_N, A_{N+1}, \ldots\}$, $N = 1, 2, \ldots$

(ii) $B_N$ *is independent of* $\{A_1, A_2, \ldots, A_N\}$, $N = 1, 2, \ldots$

*Then* $P\{A_N \bigcap B_N \text{ i.o.}\} > 0$.

This Lemma under the condition (i) is proved in Klass [6], while for the proof under (ii) we refer to Lemma 3.1 and its proof in [3].

To show Lemma 6, observe that $B_N$ defined by (6.2) is independent of $\{A_N, A_{N+1}, \ldots\}$ defined by (6.1). By Lemma 5 we have $P\{A_N \text{ i.o.}\} = 1$ and it is easy to see that $\liminf_{N\to\infty} P\{B_N\} > 0$. Hence applying Lemma 7(i) we get $P\{A_N \bigcap B_N \text{ i.o.}\} > 0$. Now let $A_N^* = A_N \bigcap B_N$. Then $C_N$ defined by (6.3) is independent of $A_1^*, A_2^*, \ldots, A_N^*$ and $\liminf_{N\to\infty} P\{C_N\} > 0$. (This can easily be seen by using our Lemma 3 together with the argument used for (3.7)-(3.8) of [4]). Hence by Lemma 7(ii), $P\{A_N \bigcap B_N \bigcap C_N \text{ i.o.}\} > 0$.

It is easy to see that $A_N \bigcap B_N \bigcap C_N$ implies

$$m_1 - 3\varepsilon < \frac{\kappa(N)}{g(\rho_N)\log_2 g(\rho_N)} < m_1 + 3\varepsilon,$$

$$m_2 - \varepsilon < \frac{N}{g(\rho_N)\log_2 g(\rho_N)} < m_2 + \varepsilon$$

and

$$m_2 - \varepsilon < \frac{L_{\rho_N + \alpha_N}}{g(\rho_N)\log_2 g(\rho_N)} < (1+\varepsilon)(m_2 + \varepsilon).$$

Moreover, for large enough $N$, $A_N$ implies

$$g(2\rho_N) \le (1+\varepsilon) g(\rho_N) < \frac{(1+\varepsilon)N}{(m_2 - \varepsilon)\log_2 g(\rho_N)} < 2N = g(\alpha_N),$$

hence $2\rho_N \le \alpha_N$. Now let $n = \rho_N/\tau$. Since $g(\rho_N)\log_2 g(\rho_N) \sim g(n)\log_2 g(n)$ we can see using Corollary 2 and 3, that $A_N \bigcap B_N \bigcap C_N$ implies

$$m_1 - \varepsilon_1 < \widetilde{f}_n(x) < m_1 + \varepsilon_1, \qquad \varepsilon_1 \le x \le \tau - \varepsilon_1$$

and

$$m_2 - \varepsilon_1 < \widetilde{f}_n(x) < m_2 + \varepsilon_1, \qquad \tau \le x \le 1$$

for some $\varepsilon_1 > 0$, which in turn implies $d(\widetilde{f}_n, h) < \varepsilon_1$. Since $\varepsilon$ and hence $\varepsilon_1$ is arbitrary, we can complete the proof of our Lemma by using the 0-1 law for $X_n$.

Now we are ready to show Property $2'$.

By choosing $\varepsilon > 0$, $\tau, m_1, m_2$ rational, and using that such functions are dense in $\mathcal{M}^{\triangle}$ with respect to the Lévy metric, one can see that there is a universal set of probability one for which $d(\widetilde{f}_n, h) < \varepsilon$ for infinitely many $n$, therefore we can find a sequence $\{n_k\}$ such that $d(\widetilde{f}_{n_k}, h) < 1/k$ implying that $\widetilde{f}_{n_k} \to h$ in the Lévy metric, i.e. $h$ is a limit point. This shows Property $2'$, and completes the proof of Theorem 2.

ACKNOWLEDGMENT

The authors are grateful to Antónia Földes for helpful advice and discussions.

## REFERENCES

[1] J. Bertoin and M. Caballero, On the rate of growth of subordinators with slowly varying Laplace exponent, *Sem. de Prob. XXIX, Lecture Notes Math*, Vol. **1612**, Springer-Verlag, Berlin, 1995, pp. 125–132.

[2] L. Breiman, Probability, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[3] E. Csáki, M. Csörgő, A. Földes and P. Révész, On the occupation time of an iterated process having no local time, *Stochastic Process. Appl.*, Vol. **70**, 1997, pp. 199–217.

[4] P. Erdős and J. Taylor, Some problems concerning the structure of random walk paths, *Acta Math. Acad. Sci. Hung.*, Vol. **11**, 1960, pp. 137–162.

[5] J.-P. Kahane, Some random series of functions, Cambridge University Press, Cambridge, 1985.

[6] M. Klass, Toward a universal law of the iterated logarithm, Part I, *Z. Wahrsch. verw. Gebiete*, Vol. **36**, 1976, pp. 165–178.

[7] M. Marcus and J. Rosen, Laws of the iterated logarithm for the local times of recurrent random walks on $Z^2$ and of Lévy processes and recurrent random walks in the domain of attraction of Cauchy random variables, *Ann. Inst. H. Poincaré Prob. Stat.*, Vol. **30**, 1994, pp. 467–499.

[8] M. Marcus and J. Rosen, Laws of the iterated logarithm for the local times of symmetric Lévy processes and recurrent random walks, *Ann. Probab.*, Vol. **22**, 1994, pp. 626–658.

[9] P. Révész and E. Willekens, On the maximal distance between two renewal epochs, *Stochastic Process. Appl.*, Vol. **27**, 1988, pp. 21–41.

*(Manuscript received May 20, 1997;*
*Revised version received February 13, 1998.)*