Commun. math. Phys. 51, 97—105 (1976)

Communications in
Mathematical
Physics

Existence of the Critical Point in ϕ^4 Field Theory*

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Abstract. We consider the ϕ^4 quantum field theory in two and three spacetime dimensions. In the single phase region the physical mass (inverse correlation length) $m(\sigma)$ decreases continuously to zero as the bare mass parameter σ approaches a critical value σ_c from above. In three dimensions the critical point σ_c is in the single phase region and the physical mass vanishes there, $m(\sigma_c)=0$.

A consequence of our results is that the critical exponent ν governing the approach to infinite correlations is bounded below (rigorously) by its classical value, 1/2.

I. Introduction and Results

In this paper we show that in the single phase region, the physical mass of the $\lambda{:}\phi^4{:}_d+\sigma{:}\phi^2{:}_d$ quantum field theory, for space-time dimension $d=2,3$, is a continuous increasing function of σ which assumes all strictly positive values. From the point of view of physics this is important since it ensures that by a suitable choice of coupling constants these theories can describe particles of any assigned mass; in short, the theory is mass renormalizable.

Let $\langle\,\rangle_\sigma$ denote expectations for the $\lambda{:}\phi^4{:}_d+\sigma{:}\phi^2{:}_d$ euclidean quantum field theory, obtained as a limit of expectations $\langle\,\rangle_{\sigma,L}$ for the half-Dirichlet theory in volume L, see [1, 2] for details. We fix the Wick ordering mass μ_0 throughout the paper. The long range order $\mathscr{L}(\sigma)$ and the energy gap $\mu(\sigma)$ are defined by:

$$\begin{aligned}\mathscr{L}(\sigma)^2&=\lim_{|r|\to\infty}\langle\phi(0)\phi(r)\rangle_\sigma,\\ \mu(\sigma)&=-\lim_{|r|\to\infty}|r|^{-1}\ln\langle\phi(0)\phi(r)\rangle_\sigma.\end{aligned}\tag{1.1}$$

The set $\Sigma\equiv\{\sigma|\mathscr{L}(\sigma)=0\}$ of zero long range order is the single phase region where these models are known to have a unique vacuum, see Simon [2]. By the *GKS* inequalities [2, 3, 4], $\mathscr{L}(\sigma)$ is decreasing in σ. Thus Σ is a proper right half-

* Supported in part by the National Science Foundation under Grant MPS74-13252

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*** Present address: Dept. of Mathematics, University of Massachusetts, Amherst, Mass., USA line, since $\mathscr{L}(\sigma)$ is known to be zero for σ sufficiently large by the cluster expansions of Glimm, Jaffe and Spencer [5], Magnen and Seneor [6], and Feldman and Osterwalder [7], while $\mathscr{L}(\sigma)$ is nonzero for σ sufficiently negative by the existence of phase transitions for these models, see Glimm et al. [8] and Fröhlich et al. [9].

The energy gap $\mu(\sigma)$ is an increasing function of σ, again by the *GKS* inequalities, and clearly $\mu(\sigma)$ vanishes outside of the single phase region. We define the critical point σ_c by:

$$\sigma_c = \sup\{\sigma \mid \mu(\sigma) = 0\}.$$

For σ in the single phase region (in particular, whenever $\mu(\sigma) > 0$), note that $\mu(\sigma)$ equals the physical mass $m(\sigma)$ which is defined for any σ by:

$$m(\sigma) = -\lim_{|r| \to \infty} |r|^{-1} \ln(\langle \phi(0)\phi(r)\rangle_\sigma - \mathscr{L}(\sigma)^2). \tag{1.2}$$

Glimm and Jaffe [10], have shown that $m(\sigma)$ is continuous in σ for $\sigma > \sigma_c + \varepsilon$, any $\varepsilon > 0$ (while their proof is for $d = 2$, it extends in a straightforward way to $d = 3$). Also, the cluster expansions [5–7] show that $m(\sigma) \uparrow \infty$ as $\sigma \uparrow \infty$.

Our principal result is a proof that for the models studied here, $m(\sigma) = \mu(\sigma) \downarrow 0$ as $\sigma \downarrow \sigma_c$. Specifically we show that for any $\sigma_2 > \sigma_c$ there is a constant such that

$$m(\sigma) = \mu(\sigma) \leqq \text{const}\,(\sigma - \sigma_c)^{1/2}, \ \sigma_c < \sigma \leqq \sigma_2. \tag{1.3}$$

Thus from the discussion in the previous paragraph, $m(\sigma)$ takes on continuously all values in $(0, \infty)$ as σ ranges over (σ_c, ∞). The bound (1.3) implies that the critical exponent ν governing the approach to infinite correlation length, defined by $m(\sigma) \sim (\sigma - \sigma_c)^\nu$, is bounded below by its classical value: $\nu \geqq 1/2$. Further bounds on critical exponents follow as in [10]. In particular, for the exponent α for the specific heat we obtain $\alpha \leqq 2\nu$ if $d = 2$ and $\alpha \leqq \nu/2$ if $d = 3$.

The bound (1.3) implies that $\mu(\sigma_c) = 0$ but this does not imply that the physical mass $m(\sigma)$ vanishes at the critical point, because of the possibility that the critical point may not be in the single phase region Σ. However in the case $d = 3$ we can show that $\sigma_c \in \Sigma$, and thus $m(\sigma_c) = 0$[1]. To show that $\sigma_c \in \Sigma$, we note that the Lehmann spectral formula provides a uniform bound, for $\sigma > \sigma_c$, on the decay of the two point function:

$$\begin{aligned} \langle \phi(0)\phi(r)\rangle_\sigma &= \textstyle\int_0^\infty d\varrho_\sigma(a)(4\pi|r|)^{-1} e^{-a|r|}, \ \sigma \in \Sigma, \ d = 3, \\ &\leqq |r|^{-1} \textstyle\int_0^\infty d\varrho_\sigma(a)(4\pi)^{-1} e^{-a}, \ |r| \geqq 1, \\ &= |r|^{-1} \langle \phi(0)\phi((1, \mathbf{0}))\rangle_\sigma \leqq |r|^{-1} \langle \phi(0)\phi((1, \mathbf{0}))\rangle_{\sigma_c}. \end{aligned} \tag{1.4}$$

Here $d\varrho_\sigma(a)$ is the spectral measure for the two-point function and we have used the monotone decrease of the two-point function as a function of σ.

The bound (1.4) extends to the critical point σ_c, showing $\sigma_c \in \Sigma$, because $\langle \phi(0)\phi(r)\rangle_\sigma$ is continuous from above in σ. To prove continuity from above in σ we note that $\langle \phi(0)\phi(r)\rangle_{\sigma, L}$ is continuous in σ, and is monotone increasing both as L increases or σ decreases, allowing the interchange of the limits $L \uparrow \infty$, $\sigma \downarrow \sigma_c$.

[1] The result $m(\sigma_c) = 0$ is also true for lattice ϕ^4 field theories and for Ising models in dimensions $d \geqq 3$ (see the Appendix)

Our results still leave open a number of questions about the nature of the critical point in $:\phi^4:_d$ theories. For $d=3$ there could be an interval $\{\sigma_0, \sigma_c)$, either open or semi-open, of values of σ lying in the single phase region but below the critical point. Thus the physical mass $m(\sigma)$ would vanish in the interval $\{\sigma_0, \sigma_c]$ as in:

$$\underset{}{\mathscr{L}(\sigma)>0}\ \underset{\sigma_0}{\{}\ m(\sigma)=0\ \underset{\sigma_c}{[}\ m(\sigma)>0 \,.$$

Behavior of $m(\sigma)$, $\mathscr{L}(\sigma)$.

Similar behavior could occur for $d=2$, with the additional possibility that the physical mass might be discontinuous at the critical point. This could occur if the long range order is discontinuous at $\sigma_c : \mathscr{L}(\sigma_c) \neq 0$. Behavior of this type actually occurs in certain Ising type models with long range interactions and is known as the Thoules's effect [11]. Finally, we are unable to say anything about the behavior of the physical mass in the multiphase region. In particular, we cannot rule out the possibility that $m(\sigma)$ is discontinuous from below at σ_0, or that there might be regions below σ_0 where $m(\sigma)$ vanishes. Such pathologies are not expected to appear in $:\phi^4:_d$ models, the anticipated picture for the critical point being that which occurs in the exactly soluble two-dimensional Ising model where $\sigma_0 = \sigma_c$, $m(\sigma_c)=0$ and the physical mass $m(\sigma)$ is continuous and strictly monotone increasing as one moves away from σ_c in either direction, see for example [12].

Glimm and Jaffe [10], were the first to study the dependence of the physical mass on σ. Using the Lebowitz inequality [2, 13, 14] they established continuity of $m(\sigma)$ above σ_c. Using related methods, Baker [15] showed the continuity of a pseudomass in lattice ϕ^4 models and in [16], Rosen showed how these ideas could be modified to prove continuity of the mass itself for ϕ^4 lattice fields in the single phase-region. This paper extends these ideas to the continuum limit for space-time dimensions $d=2, 3$.

In Section II we define a pseudomass $\mu^{\urcorner}(\sigma)$ (more precisely it is a pseudo-energy gap) as the limit of finite volume quantities $\mu^{\urcorner}(\sigma, L)$. The $\mu^{\urcorner}(\sigma, L)$ are defined so as to be always strictly positive, even for $\sigma < \sigma_c$. In section III we relate the pseudomass and energy gap by bounds of the form

$$\mu^{\urcorner}(\sigma) \leqq \mu(\sigma) \leqq \text{const}\, \mu^{\urcorner}(\sigma)\,. \tag{1.5}$$

In Section IV we show that $\mu^{\urcorner}(\sigma, L)$ is Lipschitz continuous in σ, using the Lebowitz inequality [2, 13, 14] and ϕ-bounds [2, 18, 19]. Heuristically, our proof amounts to obtaining a bound of the form:

$$\frac{d}{d\sigma}\mu^{\urcorner}(\sigma, L) \leqq \text{const}\, \mu^{\urcorner}(\sigma, L)^{-d-1}\,, \tag{1.6}$$

with the constant uniformly bounded in σ, L for σ in compact sets. Since $\mu^{\urcorner}(\sigma, L) > 0$, such a bound makes sense, and we may integrate (1.6) to obtain Lipschitz continuity of $\mu^{\urcorner}(\sigma, L)$ and thus also of $\mu^{\urcorner}(\sigma)$. The bound (1.5) implies that $\mu^{\urcorner}(\sigma)=0$, $\sigma<\sigma_c$, and continuity then implies $\mu^{\urcorner}(\sigma_c)=0$. Therefore, again using (1.5), $\mu(\sigma_c)=0$ and $\mu(\sigma)\downarrow 0$ as $\sigma \downarrow \sigma_c$. Thus all of our results follow from the continuity of $\mu^{\urcorner}(\sigma)$. The bound (1.3) is proved in [10], for $d=2$, under the assumption that $m(\sigma)\to 0$ as $\sigma\to\sigma_c$ (which we have now proved). The proof given in [10] applies also to the case $d=3$.

II. Definition and Properties of the Pseudomass

Let $\sigma_1<\sigma_c<\sigma_2$ be fixed numbers on either side of σ_c. We will use σ_1, σ_2 as reference points and throughout the paper we assume $\sigma_1\leqq\sigma\leqq\sigma_2$. By the *GKS* inequalities [2–4],

$$0<\langle\phi(0)\phi((1/2,\mathbf{0}))\rangle_\sigma\leqq\phi(0)\phi((1/2,\mathbf{0}))\rangle_{\sigma_1}\equiv A^2<\infty\,. \tag{2.1}$$

It is convenient to normalize the field by $\psi(r)\equiv\phi(r)(1+A)^{-1}$ so that

$$\begin{aligned}&0<\langle\psi(0)\psi(r)\rangle_{\sigma,L}\leqq\langle\psi(0)\psi(r)\rangle_\sigma<1,\ |r|\geqq 1/2,\ \sigma\geqq\sigma_1\,,\\ &\mu(\sigma)=-\lim_{|r|\to\infty}|r|^{-1}\ln\langle\psi(0)\psi(r)\rangle\,,\end{aligned} \tag{2.2}$$

where we have used the monotonity properties of the two-point function in L, $|r|$.

We define the pseudomass $\mu\tilde{}(\sigma)$ as the limit of finite volume quantities $\mu\tilde{}(\sigma, L)$ which are monotone decreasing in the volume $|L|\geqq 1$ of squares L centered at the origin in spacetime:

$$\tilde{\mu}\,(\sigma)=\lim_{L\to\infty}\mu\tilde{}(\sigma, L)=\inf_L\mu\tilde{}(\sigma, L)\,. \tag{2.3}$$

For each pair of points r, $s\in L$, $|r-s|\geqq 1$, we define $\mu\tilde{}(\sigma, L, r, s)$ to be the unique solution $\mu\tilde{}$ of the equation

$$e^{-\tilde{\mu}|r-s|}(1+(\mu\tilde{}|r-s|)^{(d+1)/2})^{-1}\equiv\langle\psi(r)\psi(s)\rangle_{\sigma,L} \tag{2.4}$$

and we define the finite volume pseudomass by

$$\mu\tilde{}(\sigma, L)=\inf\{\mu\tilde{}(\sigma, L, r, s)|r, s\in L, |r-s|\geqq 1\}\,. \tag{2.5}$$

That (2.4) has a unique, strictly positive solution follows from the fact that the strictly monotone decreasing function $e^{-x}(1+x^{(d+1)/2})^{-1}$ ranges over (0, 1) as x ranges over $(0,\infty)$, while the right side of (2.3) lies in (0, 1) by (2.2). The monotone decreasing property of $\mu\tilde{}(\sigma, L, r, s)$ and $\mu\tilde{}(\sigma, L)$ in L follows since $\langle\psi(r)\psi(s)\rangle_{\sigma,L}$ is monotone increasing in L. Similarly, $\mu\tilde{}(\sigma)$, $\mu\tilde{}(\sigma, L)$, $\mu\tilde{}(\sigma, L, r, s)$ are all monotone increasing in σ since $\langle\psi(r)\psi(s)\rangle_{\sigma,L}$ is monotone decreasing in σ. We note that by the continuity of $\langle\psi(r)\psi(s)\rangle_{\sigma,L}$ in r, s there are $r_{\sigma,L}, s_{\sigma,L}\in L$ with

$$\mu\tilde{}(\sigma, L)=\mu\tilde{}(\sigma, L, r_{\sigma,L}, s_{\sigma,L})>0\,.$$

We will later use the following result:

Lemma 1. *$\mu\tilde{}(\sigma, L)$ is continuous from below in σ.*

Proof. Let $\sigma_i\uparrow\sigma$. By compactness, there is a subsequence σ'_j of σ_i and a pair of points r, $s\in L$ with $r_{\sigma'_j,L}\to r$, $s_{\sigma'_j,L}\to s$, $|r-s|\geqq 1$. Thus by the continuity of $\mu\tilde{}(\sigma,L,r,s)$ in σ, r, s:

$$\mu\tilde{}(\sigma'_j, L)=\mu\tilde{}(\sigma_j, L, r_{\sigma'_j,L}, s_{\sigma'_j,L})\xrightarrow[j\to\infty]{}\mu\tilde{}(\sigma, L, r, s)\geqq\mu\tilde{}(\sigma, L)\,.$$

But $\mu\tilde{}(\sigma'_j, L)\leqq\mu\tilde{}(\sigma', L)$ by monotonicity in σ; continuity from below in σ follows.

III. Comparison of Energy Gap and Pseudomass

We relate the properties of the pseudomass to the energy gap $\mu(\sigma)$ by the following result:

Theorem 2. *For all $\sigma \geqq \sigma_1$, $\mu\check{}(\sigma) \leqq \mu(\sigma) \leqq (d+3)\mu\check{}(\sigma)$.*

Proof. To establish the left-hand inequality, we fix r, $|r| \geqq 1$. For any $L \ni r$, we have by (2.4) and $\mu\check{}(\sigma, L) \geqq \mu\check{}(\sigma)$:

$$-|r|^{-1} \ln \langle \psi(0)\psi(r)\rangle_{\sigma,L} \geqq \mu\check{}(\sigma) + |r|^{-1} \ln(1 + (\mu\check{}(\sigma)|r|)^{(d+1)/2}).$$

Since the right-hand side is independent of L,

$$-|r|^{-1} \ln \langle \psi(0)\psi(r)\rangle_{\sigma} \geqq \mu\check{}(\sigma) + |r|^{-1} \ln(1 + (\mu\check{}(\sigma)|r|)^{(d+1)/2}).$$

The left-hand inequality of Theorem 2 follows on taking $|r| \to \infty$.

To establish the right-hand inequality, we prove below that

$$\mu(\sigma) \leqq -2|r|^{-1} \ln \langle \psi(0)\psi(r)\rangle_{\sigma}, \quad |r| \geqq 1. \tag{3.1}$$

Thus for each L we have by (2.4) and translation invariance:

$$\begin{aligned}
\mu(\sigma) &\leqq -2|r_{\sigma,L} - s_{\sigma,L}|^{-1} \ln \langle \psi(r_{\sigma,L})\psi(s_{\sigma,L})\rangle_{\sigma} \\
&\leqq -2|r_{\sigma,L} - s_{\sigma,L}|^{-1} \ln \langle \psi(r_{\sigma,L})\psi(s_{\sigma,L})\rangle_{\sigma,L} \\
&= 2[\mu\check{}(\sigma, L) + |r_{\sigma,L} - s_{\sigma,L}|^{-1} \ln(1 + (\mu\check{}(\sigma, L)|r_{\sigma,L} - s_{\sigma,L}|)^{(d+1)/2})] \\
&\leqq (d+3)\mu\check{}(\sigma, L),
\end{aligned}$$

where we have used $\ln(1+x^a) \leqq \ln(1+x)^a \leqq ax$, $a \geqq 1$, $x \geqq 0$. The right-hand inequality of Theorem 2 follows on letting $L \to \infty$.

To prove the bound (3.1), we introduce test-functions $f(\cdot) \varepsilon C_0^\infty(R^d)$, with supports in the sphere of radius 1/4, and we define smeared fields by $\psi_f(r) = \int d^d x f(x-r)\psi(x)$. Thus by translation invariance and Osterwalder-Schrader positivity [17]

$$\begin{aligned}
\langle \psi_f(0)\psi_f(r)\rangle_{\sigma} &= \langle \psi_f(-n/2)\psi_f(r-n/2)\rangle_{\sigma}, \quad n \equiv (1/2, \mathbf{0}), \\
&\leqq \langle \psi_f(0)\psi_f(n)\rangle_{\sigma}^{1/2} \langle \psi_f(0)\psi_f(2r-n)\rangle_{\sigma}^{1/2} \\
&\leqq \langle \psi_f(0)\psi_f(n)\rangle_{\sigma} \lim_{l \to \infty} \langle \psi_f(0)\psi_f(2^l(r-n)+n)\rangle_{\sigma}^{1/2^l} \\
&= \langle \psi_f(0)\psi_f(n)\rangle_{\sigma} e^{-\mu(\sigma)|r-n|},
\end{aligned} \tag{3.2}$$

where in the second to last step, we have iterated the previous inequality infinitely often, while in the last step we have used the definition (2.2) of $\mu(\sigma)$. The bound (3.1) now follows from (3.2) by choosing a sequence $f(\cdot) \to \delta^{(d)}(\cdot)$, and noting that for $|r| \geqq 1$, $|r-n| \geqq |r|/2$ and that $\langle \psi(0)\psi(n)\rangle_{\sigma} \leqq 1$ by (2.2).

IV. Continuity of the Pseudomass

Theorem 3. *For any σ_1, σ_2 there is a constant $k(\sigma_1, \sigma_2)$ with:*

$$0 \leqq \mu\check{}(\sigma')^{d+2} - \mu\check{}(\sigma)^{d+2} \leqq k(\sigma' - \sigma), \quad \sigma_1 \leqq \sigma \leqq \sigma' \leqq \sigma_2. \tag{4.1}$$

Proof. It is sufficient to prove (4.1) with $\mu\check{}(\sigma)$ replaced by $\mu\check{}(\sigma, L)$ and a constant k independent of L. We will show below that there is a constant c, independent of L and of $\sigma, \sigma_1 \leqq \sigma \leqq \sigma_2$, such that

$$\frac{d}{d\sigma} \mu\check{}(\sigma, L, r, s)^{d+2}|_{r_{\sigma,L}, s_{\sigma,L}} \leqq c\,. \tag{4.2}$$

Thus for each $\sigma \in [\sigma_1, \sigma_2)$, there is a $\sigma''(\sigma, L) > \sigma$ with

$$\mu\check{}(\sigma', L, r_{\sigma,L}, s_{\sigma,L})^{d+2} - \mu\check{}(\sigma, L, r_{\sigma,L}, s_{\sigma,L})^{d+2} \leqq (c+1)(\sigma' - \sigma)\,, \tag{4.3}$$

for $\sigma \leqq \sigma' \leqq \sigma''$. Since $\mu\check{}(\sigma', L) \leqq \mu\check{}(\sigma', L, r_{\sigma,L}, s_{\sigma,L})$, with equality when $\sigma' = \sigma$, (4.3) implies that for $\sigma \leqq \sigma' \leqq \sigma''$:

$$\mu\check{}(\sigma', L)^{d+2} - \mu\check{}(\sigma, L)^{d+2} \leqq (c+1)(\sigma' - \sigma)\,. \tag{4.4}$$

Let $I_{\sigma,L}$ denote the maximal interval in $[\sigma, \sigma_2]$ containing σ and such that (4.4) is valid for $\sigma' \in I_{\sigma,L}$. To complete the proof of Theorem 3, we need only show that $I_{\sigma,L} \equiv [\sigma, \sigma_2]$ for all σ, L. By Lemma 1, $I_{\sigma,L}$ is closed: $I_{\sigma,L} = [\sigma, \sigma\hat{}]$ for some $\sigma\hat{} = \sigma\hat{}(\sigma, L)$. If $\sigma\hat{} \neq \sigma_2$, then $I_{\sigma\hat{},L} \not\subset I_{\sigma,L}$, and yet for $\sigma' \in I_{\sigma\hat{},L}$:

$$\begin{aligned}
&\mu\check{}(\sigma',L)^{d+2} - \mu\check{}(\sigma,L)^{d+2} = \mu\check{}(\sigma',L)^{d+2} - \mu\check{}(\sigma\hat{},L)^{d+2} + \mu\check{}(\sigma\hat{},L)^{d+2} - \mu\check{}(\sigma,L)^{d+2} \\
&\leqq (c+1)(\sigma' - \sigma\hat{}) + (c+1)(\sigma\hat{} - \sigma) = (c+1)(\sigma' - \sigma)\,,
\end{aligned}$$

which implies that $I_{\sigma\hat{},L} \subseteq I_{\sigma,L}$. The contradiction forces the conclusion that $\sigma\hat{} = \sigma_2$.

It remains to prove the bound (4.2). Differentiating the defining relation (2.4) for $\mu\check{} \equiv \mu\check{}(\sigma, L, r, s)$ with respect to σ we obtain:

$$\begin{aligned}
&|r-s|[1 + 2^{-1}(d+1)(\mu\check{}|r-s|)^{(d-1)/2}(1 + (\mu\check{}|r-s|)^{(d+1)/2})^{-1}]\langle\psi(r)\psi(s)\rangle_{\sigma,L} \frac{d\mu\check{}}{d\sigma} \\
&= -\frac{d}{d\sigma}\langle\psi(r)\psi(s)\rangle_{\sigma,L} \\
&= \int_L d^d t \{\langle\psi(r)\psi(s):\phi^2(t):\rangle_{\sigma,L} - \langle\psi(r)\psi(s)\rangle_{\sigma,L}\langle:\phi^2(t):\rangle_{\sigma,L}\} \\
&\leqq (1+A)^2 \int_L d^d t \langle\psi(r)\psi(t)\rangle_{\sigma,L}\langle\psi(t)\psi(s)\rangle_{\sigma,L}\,,
\end{aligned}$$

where we have used the Lebowitz inequality [2, 13, 14] in the last step, and $(1+A)$ is the normalization factor relating ϕ and ψ, see (2.1). Bounding below by 1 the term in rectangular brackets, we have:

$$\frac{d\mu\check{}}{d\sigma} \leqq (1+A)^2 |r-s|^{-1} \langle\psi(r)\psi(s)\rangle_{\sigma,L}^{-1} \int_L d^d t \langle\psi(r)\psi(t)\rangle_{\sigma,L}\langle\psi(t)\psi(s)\rangle_{\sigma,L}\,. \tag{4.5}$$

We decompose the region of integration into four parts: $L = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ and we denote the corresponding contributions to (4.5) by $D_{\mathrm{I}}, \ldots, D_{\mathrm{IV}}$. Here $\mathrm{I} = \{t \in L : |t-r|, |t-s| \geqq 1\}$, $\mathrm{II} = \{t \in L : |t-s| \geqq 1 > |t-r|\}$, $\mathrm{III} = \{t \in L : |t-r| \geqq 1 > |t-s|\}$ and $\mathrm{IV} = \{t \in L : |t-r|, |t-s| < 1\}$. The derivative in (4.2) is to be evaluated at the point $r = r_{\sigma,L}$, $s = s_{\sigma,L}$ and in the following we set r, s equal to these values.

In region I, using the definition (2.4) and the fact that $\mu\check{}(\sigma, L) = \mu\check{}(\sigma, L, r, s)$, we obtain

$$\begin{aligned}
D_{\mathrm{I}} &\leqq (1+A)^2 |r-s|^{-1}(1+(\mu\check{}(\sigma,L)|r-s|)^{(d+1)/2}) \int d^d t e^{-\tilde{\mu}(\sigma,L)\{|r-t|+|t-s|-|r-s|\}} \\
&\quad (1+(\mu\check{}(\sigma,L)|r-t|)^{(d+1)/2})^{-1}(1+(\mu\check{}(\sigma,L)|t-s|)^{(d+1)/2})^{-1} \\
&\leqq c_1|r-s|^{(d-1)/2}\mu\check{}(\sigma,L)^{-d-1}\int d^d t|r-t|^{-(d+1)/2}|t-s|^{-(d+1)/2} \\
&\leqq c_1|r-s|^{(d-3)/2}\mu\check{}(\sigma,L)^{-d-1}\int d^d t(|t|\,|t-(1,\mathbf{0})|)^{-(d+1)/2} \\
&\leqq c_2\mu\check{}(\sigma,L)^{-d-1}\,.
\end{aligned}$$

Here and in what follows, all constants c_i are uniform in L and in σ, for $\sigma \in [\sigma_1, \sigma_2]$. In particular using monotonicity in σ, L (and identifying L with $|L|$), we may choose

$$c_1 = (1+A)^2(1+\mu\check{}(\sigma_2, 1)^{(d+1)/2}),\; c_2 = c_1 \int d^d t(|t|\,|t-(1,\mathbf{0})|)^{-(d+1)/2}\,.$$

For region II, we have the bound:

$$\begin{aligned}
D_{\mathrm{II}} &\leqq (1+A)^2|r-s|^{-1}(1+(\mu\check{}(\sigma,L)|r-s|)^{(d+1)/2})\int_{\mathrm{II}} d^d t e^{-\tilde{\mu}(\sigma,L)\{|t-s|-|r-s|\}} \\
&\quad (1+(\mu\check{}(\sigma,L)|t-s|)^{(d+1)/2})^{-1}\langle\psi(r)\psi(t)\rangle_{\sigma,L} \\
&\leqq c_3 \int_{|t|<1} d^d t\langle\psi(0)\psi(t)\rangle_\sigma\,,
\end{aligned} \tag{4.6}$$

where $c_3 = (1+A)^2 2^{(d+1)/2} e^{\tilde{\mu}(\sigma_2, 1)}$ and we have noted that

$$|t-s|-|r-s| \geqq -|r-t| \geqq -1,\; |r-s| \leqq 1+|t-s| \leqq 2|t-s|\,.$$

An identical estimate applies to D_{III}, while for region IV we note that either $|r-t|$ or $|t-s|$ is greater than 1/2 since $|r-s| \geqq 1$. Thus by (2.1) either $\langle\psi(r)\psi(t)\rangle_\sigma$ or $\langle\psi(t)\psi(s)\rangle_\sigma$ is bounded by 1 so that:

$$\begin{aligned}
D_{\mathrm{IV}} &\leqq (1+A)^2|r-s|^{-1}(1+(\mu\check{}(\sigma,L)|r-s|)^{(d+1)/2})e^{\tilde{\mu}(\sigma,L)|r-s|}\int_{|t|\leqq 1} d^d t\langle\psi(0)\psi(t)\rangle_\sigma\,, \\
&\leqq c_4 \int_{|t|\leqq 1} d^d t\langle\psi(0)\psi(t)\rangle_\sigma\,,
\end{aligned} \tag{4.7}$$

where $c_4 = (1+A)^2(1+(2\mu\check{}(\sigma_2, 1)^{(d+1)/2})e^{2\tilde{\mu}(\sigma_2, 1)}$ and we have noted that region IV is empty unless $|r-s| \leqq 2$. To bound the integrals in (4.6), (4.7) we observe that by translation invariance:

$$\begin{aligned}
&\int_{|t|\leqq 1} d^d t\langle\psi(0)\psi(t)\rangle_\sigma \leqq (2\pi)^{-1}\int_{|s|\leqq 1} d^d s \int_{|t|\leqq 1} d^d t\langle\psi(s)\psi(t+s)\rangle_\sigma\,. \\
&\leqq \int_{|s|\leqq 2} d^d s \int_{|r|\leqq 2} d^d r\langle\psi(s)\psi(r)\rangle_\sigma,\; r=t+s\,, \\
&= \langle\psi(\chi_2)^2\rangle_\sigma \leqq \langle\psi(\chi_2)^2\rangle_{\sigma_1} \equiv c_5\,,
\end{aligned} \tag{4.8}$$

where $\psi(\chi_2)$ denotes the field ψ smeared with the characteristic function χ_2 of the circle (sphere) of radius 2, and in the last step we have used a ϕ-bound [2, 18, 19]. Combining (4.5), (4.6), (4.7) we obtain, with $c_6 = (2c_3 + c_4)c_5$,

$$D_{\mathrm{II}} + D_{\mathrm{III}} + D_{\mathrm{IV}} \leqq c_6 \leqq c_7 \mu\check{}(\sigma, L)^{-d-1}\,,$$

where $c_7 = c_6 \mu\check{}(\sigma_2, 1)^{d+1}$, which completes the proof of (4.2).

Acknowledgement. We would like to thank James Glimm for a number of useful conversations.

Appendix

Theorem. *In lattice ϕ^4 field theories or in Ising models, the critical point is in the single phase region for dimension $d \geqq 3$.*

Proof. Without loss of generality we consider the lattice spacing to be one. By a result of Fröhlich et al. [9], the two-point function in momentum space has the representation:

$$S_\sigma(p) = c_\sigma \delta^{(d)}(p) + f_\sigma(p) \tag{A.1}$$

where for $\sigma_1 \leqq \sigma_c \leqq \sigma_2$ there is a constant a with

$$0 \leqq f_\sigma(p) \leqq a/p^2, \sigma_1 \leqq \sigma \leqq \sigma_2 . \tag{A.2}$$

For δ^{-1} integral, let $h_\delta(x) \equiv (2\pi)^{d/2}(1+2\delta^{-1})^{-d}\chi_\delta(x)$ where $\chi_\delta(x)$ is the characteristic function of $\{x \in Z^d : |x_i| \leqq \delta^{-1}, i=1, \ldots, d\}$. The lattice fourier transform of h_δ satisfies, for $d \geqq 3$,

$$h\tilde{}_\delta(0) = 1, \quad \int_{|p_i| \leqq \pi} d^dp p^{-2} |h\tilde{}_\delta(p)| \to 0 \quad \text{as} \quad \delta \to 0 . \tag{A.3}$$

Thus from (A.1)–(A.3), we see that the constant c_σ is given by:

$$c_\sigma = \lim_{\delta \to 0} c_{\sigma,\delta} \equiv \lim_{\delta \to 0} \int_{|p_i| \leqq \pi} d^dp S_\sigma(p) h\tilde{}_\delta(p) .$$

By definition, $c_\sigma = 0$ for $\sigma > \sigma_c$ and we wish to prove that $c_{\sigma_c} = 0$, which is equivalent to showing that $c_{\sigma_c,\delta} \to 0$ as $\delta \to 0$. Assuming for the moment that $c_{\sigma,\delta}$ is continuous from above in σ, it is therefore sufficient to prove that $c_{\sigma,\delta}$ converges to zero as $\delta \to 0$, uniformly in $\sigma_c < \sigma \leqq \sigma_2$. This follows immediately from the bound:

$$c_{\sigma,\delta} = \int d^dp S_\sigma(p) h\tilde{}_\delta(p) \leqq a \int d^dp p^{-2} |h\tilde{}_\delta(p)|, \sigma_c < \sigma \leqq \sigma_2 .$$

To prove the assumed upper semi-continuity of $c_{\sigma,\delta}$ in σ, note that

$$c_{\sigma,\delta} = \int d^dp S_\delta(p) h\tilde{}_\delta(p) = \sum_{x \in Z^d} S_\sigma(x) h_\delta(x)$$

$$= \lim_{L \to \infty} \lim_{\sigma' \to \sigma+} \sum_{x \in L} S_{\sigma',L}(x) h_\delta(x) . \tag{A.4}$$

Since $h_\delta(x)$ is positive with $S_{\sigma',L}$ positive and monotone increasing both as $\sigma' \to \sigma+$ and as $L \to \infty$, the two limits in (A.4) may be interchanged, proving the required upper-semi-continuity in σ.

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Communicated by A. S. Wightman

Received March 29, 1976; in revised form May 27, 1976