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Laws of the iterated logarithm for the local times of recurrent random walks on Z^2 and of Lévy processes and Random walks in the domain of attraction of Cauchy random variables

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ABSTRACT. – Laws of the iterated logarithm are obtained for the number of visits of a recurrent symmetric random walk on Z^2 to a point in it's state space and for the difference of the number of visits to two points in it's state space. Following convention the number of visits is called the local time of the random walk. Laws of the iterated logarithm are also obtained for the local time of a symmetric Lévy process, at a fixed point in it's space, as time goes to infinity and for the difference of the local times at two points in it's state space for Lévy processes which at a fixed time are in the domain of attraction of a Cauchy random variable. Similar results are obtained for the local times of symmetric recurrent random walks on Z^1 which are in the

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domain of attraction of Cauchy random variables. These results are related by the fact that the truncated Green's functions of all these processes are slowly varying at infinity and follow from one basic theorem.

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RÉSUMÉ. – Nous montrons des lois du logarithme itéré pour le nombre de fois qu'une marche aléatoire dans Z^2 visite un point donné et pour la différence entre le nombre de fois que cette marche visite un point donné et un autre point, lui aussi donné. Le nombre de fois qu'une marche aléatoire visite un point est le temps local de cette marche évalué en ce point. Nous montrons aussi des lois du logarithme itéré pour le temps local d'un processus de Lévy évalué en un point donné de l'espace quand le temps diverge et pour les accroissements des temps locaux lorsque le processus de Lévy est à temps fixé, dans le domaine d'attraction de la loi de Cauchy. Tous les processus considérés ont une fonction de Green tronquée qui varie lentement à l'infini et les résultats précédents sont des conséquences d'un théorème général.

1. INTRODUCTION

In this paper, which is a sequel to [MR1], we obtain first and second order laws of the iterated logarithm for the local times of a large class of symmetric Lévy processes and recurrent random walks that were not considered in [MR1], including all recurrent random walks on a two dimensional lattice which are in the domain of attraction of a Gaussian random variable. Since the study of such random walks was the primary motivation for this paper we will begin by stating this result.

Let $X = \{X_n, n \geqq 0\}$ be a symmetric random walk on the integer lattice Z^d, i.e.

$$X_n = \sum_{i=1}^{n} Y_i$$

where the random variables $\{Y_i, i \geqq 1\}$ are symmetric, independent and identically distributed with values in Z^d. We assume for convenience that the law of Y_1 is not supported on a proper subgroup of Z^d. The local time $L = \{L_n^y, (n, y) \in N \times Z^d\}$ of X is simply the family of random variables $L_n^y = \{\text{number of times } j : X_j = y, 0 \leqq j \leqq n\}$. X has symmetric transition probabilities $p_n(x-y) \equiv p_n(x, y)$. We assume that the truncated Green's function

$$g(n) = \sum_{k=0}^{n} p_k(0) \tag{1.1}$$

satisfies

$$\lim_{n \to \infty} g(n) = \infty. \tag{1.2}$$

Condition (1.2) is equivalent to X being recurrent and implies that d can only equal one or two. Let

$$\phi(\lambda) = \mathrm{E}e^{i\langle \lambda, \mathrm{Y}_1 \rangle}, \qquad \lambda \in [-\pi, \pi]^d, \qquad d = 1, 2 \tag{1.3}$$

and note that

$$p_n(0) = \left(\frac{1}{2\pi}\right)^d \int_{[-\pi, \pi]^d} \phi^n(\lambda)\, d\lambda, \qquad d = 1, 2. \tag{1.4}$$

For $x \in \mathrm{Z}^d$, $d = 1, 2$, define

$$\begin{aligned} \sigma^2(x) &= \sum_{n=0}^{\infty} (p_n(0) - p_n(x)) \\ &= \left(\frac{1}{2\pi}\right)^d \int_{[-\pi, \pi]^d} \frac{1 - \cos\langle \lambda, x \rangle}{1 - \phi(\lambda)}\, d\lambda. \end{aligned} \tag{1.5}$$

The following result is for random walks on Z^2:

THEOREM 1.1. – *Let* X *be a recurrent symmetric random walk with values in* Z^2 *as defined above and let* $g(n)$ *and* $\sigma^2(x)$ *be as defined in* (1.1) *and* (1.5). *Assume that* g *is slowly varying at infinity and let* $\{\mathrm{L}_n^y, (n, y) \in \mathrm{N} \times \mathrm{Z}^2\}$ *denote the local times of* X. *Then*

$$\overline{\lim_{n \to \infty}} \frac{\mathrm{L}_n^0}{g(n) \log\log g(n)} = 1 \quad a.s. \tag{1.6}$$

and for any $x \in \mathrm{Z}^2$

$$\overline{\lim_{n \to \infty}} \frac{\mathrm{L}_n^0 - \mathrm{L}_n^x}{g^{1/2}(n) \log\log g(n)} = \sqrt{2}\, \sigma(x) \quad a.s. \tag{1.7}$$

Furthermore (1.7) *also holds with the numerator on the left-hand side replaced by* $\sup_{0 \leq k \leq n} \mathrm{L}_k^0 - \mathrm{L}_k^x$.

The condition that g is slowly varying at infinity is satisfied if Y_1 is in the domain of attraction of an R^2 valued Gaussian random variable.

We refer to (1.6) as a first order law because the local time appears alone and to (1.7) as a second order law because it involves the difference of the local times at two points in the state space. For the simple random walk on Z^2

$$g(n) \sim \frac{\log n}{\pi} \quad \text{as} \quad n \to \infty.$$

In this case (1.6) was obtained by Erdös and Taylor [ET]. It was this result which motivated our work. We do not think that (1.7) was known even for a simple random walk. In general, one can find a random walk on Z^2 for which the denominators in (1.6) and (1.7) grow as slowly as desired.

Theorem 1.1 is a corollary of a larger study of laws of the iterated logarithm for the local times of recurrent symmetric Lévy processes and random walks for which the truncated Green's function is slowly varying at infinity. Only real valued Lévy processes have local times. However, random walks on Z^1 and Z^2 can have a slowly varying truncated Green's function. In order to present our more general results we introduce some notation for Lévy processes. (We will often use the same symbols as we did when considering random walks since it will always be clear to which processes we are referring.)

Let $X = \{X(t),\ t \in R^+\}$ be a symmetric Lévy process and set

$$E^0 \exp(i\lambda X(t)) = \exp(-t\psi(\lambda)). \tag{1.8}$$

X has a local time if and only if $(\gamma + \psi(\lambda))^{-1} \in L^1(R^+)$ for some $\gamma > 0$, and consequently for all $\gamma > 0$, (*see* [K]). We shall assume somewhat more than this, namely that

$$\int_0^\infty \frac{\log(1+\lambda)}{1+\psi(\lambda)}\, d\lambda < \infty. \tag{1.9}$$

We denote the local time of X by $\{L_t^y,\ (t, y) \in R^+ \times R\}$, which we normalize by setting

$$E^x \left(\int_0^\infty e^{-t}\, dL_t^y \right) = \int_0^\infty e^{-t}\, p_t(x-y)\, dt$$

where $p_t(x-y) \equiv p_t(x, y)$ is the transition probability density of X. We assume that

$$p_t(0) \text{ is regularly varying at infinity with index minus one} \tag{1.10}$$

and that the truncated Green's function

$$g(t) = \int_0^t p_s(0)\, ds \tag{1.11}$$

which, because of (1.10), is slowly varying at infinity, satisfies

$$\lim_{t \to \infty} g(t) = \infty. \tag{1.12}$$

Condition (1.12) is equivalent to X being recurrent. Since

$$p_t(0) = \frac{1}{\pi} \int_0^\infty e^{-t\psi(\lambda)}\, d\lambda$$

condition (1.10) is equivalent to $\psi(\lambda)$ being regularly varying at zero with index 1 (*see* e.g. Remark 2.7 in [MR1] which is also valid for $\beta = 1$) or, equivalently, to X (1) being in the domain of attraction of a Cauchy random variable. In [MR1] we considered the case where $p_t(0)$ was regularly varying at infinity with index $-1/\alpha$, $1 < \alpha \leqq 2$, which is equivalent to X (1) being in the domain of attraction of an α-stable random variable. The approach of [MR1] extends to some cases in which X (1) is in the domain of attraction of an 1-stable random variable but we also need a new approach to handle all the cases considered in Theorem 1.2.

Let

$$\sigma^2(x) = \int_0^\infty (p_t(0) - p_t(x))\, dt = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos \lambda x}{\psi(\lambda)}\, d\lambda \tag{1.13}$$

We see from the last integral that $\sigma^2(x)$ is finite.

Besides being slowly varying at infinity we will require that the function g defined in (1.11) satisfies at least one of the following conditions:

$$\lim_{t \to \infty} \frac{g(t/\log \log g(t))}{g(t)} = 1 \tag{1.14}$$

$$\overline{\lim_{t \to \infty}} \frac{g(t/\log g(t))}{g(t)} < 1 \tag{1.15}$$

$$\frac{1}{g(t)} \int_1^t \frac{g(y)}{y}\, dy \leqq C \frac{g(t)}{tg'(t)} \quad \text{for all } t \text{ sufficiently large} \tag{1.16}$$

Note that (1.14) is satisfied by the usual examples one gives for slowly varying functions (like powers of logarithms) and that (1.14) and (1.15) together cover all slowly varying functions with mild smoothness properties. Clearly the conditions may overlap. Also we will show in Section 3 that (1.15) and (1.16) can be realized by many examples.

Our main result for Lévy processes is the following;

THEOREM 1.2. – *Let* X *be a recurrent symmetric Lévy process for which* (1.9) *and* (1.10) *are satisfied and for which the function* g *defined in* (1.11) *satisfies at least one of the conditions* (1.14)-(1.16). *Let* $\{L_t^y, (t, y) \in R^+ \times R\}$ *denote the local times of* X. *Then*

$$\overline{\lim_{t\to\infty}} \frac{L_t^0}{g(t/\log\log g(t))\log\log g(t)} = 1 \quad a.s. \tag{1.17}$$

and for any $x \in R$

$$\overline{\lim_{t\to\infty}} \frac{L_t^0 - L_t^x}{g^{1/2}(t/\log\log g(t))\log\log g(t)} = \sqrt{2}\,\sigma(x) \quad a.s. \tag{1.18}$$

Furthermore (1.18) *also holds with the numerator on the left-hand side replaced by* $\sup_{0\leqq u\leqq t} L_u^0 - L_u^x$.

Theorem 1.2. also holds for symmetric random walks with the obvious modifications.

THEOREM 1.3. – *Let* X *be a recurrent symmetric random walk with values in* Z^d, $d = 1, 2$, *as defined above and let* $p(n)$, $g(n)$ *and* $\sigma^2(x)$ *be as defined for random walks in* (1.1), (1.4) *and* (1.5). *Assume that* $p(n)$ *satisfies* (1.10) *with* t *replaced by* n. *(This implies that* $g(n)$ *is slowly varying at infinity). Assume also that* $g(n)$ *satisfies at least one of the conditions* (1.14)-(1.16) *with* t *replaced by* n. *Let* $\{L_n^y, (n, y) \in N \times Z^d\}$ *denote the local times of* X. *Then* (1.17), (1.18) *and the comment immediately following it of Theorem* 1.2. *hold as stated except that the limit superior and the supremum are taken over the integers.*

Set

$$\psi(\lambda) = 1 - \phi(\lambda) \tag{1.19}$$

where $\phi(\lambda)$ is given in (1.3). In one dimension, *i.e.* for $\lambda \in [-\pi, \pi]$, we see by (1.4) and (1.19) that condition (1.10), with t replaced by n, is equivalent to $\psi(\lambda)$ being regularly varying at zero with index 1 or, equivalently, to X (1) being in the domain of attraction of a Cauchy random variable.

Theorem 1.1 is a corollary of Theorem 1.3. It has a simpler statement because (1.14), with t replaced by n, is always satisfied by symmetric random walks on Z^2 when g is slowly varying at infinity.

As far as we know, aside from the result of Erdös and Taylor just mentioned, all the results in Theorems 1.1-1.3 are new. However, we should mention that (1.17), for Lévy processes but without the constant evaluated, follows from [FP], as is pointed out in Theorem 6.9 [F]. All the other earlier work that we are aware of requires that g be regularly varying at infinity with index greater than zero, or equivalent conditions. *See* [MR1] for further discussion of prior results.

This paper is an extension of [MR1]. However the results in [MR1] are not expressed in terms of the truncated Green's function but in terms of the α-potential density of the Lévy process at zero, considered as a function of α, and similarly for random walks. The results in [MR1], which are analogous to Theorems 1.2 and 1.3, are given in Theorem 1.4 in terms of the truncated Green's functions. In order to make Theorem 1.4 more meaningful note that if ψ is regularly varying at zero with index $1 < \beta \leqq 2$ then g is regularly varying at infinity with index $1/\overline{\beta}$, where $1/\beta + 1/\overline{\beta} = 1$. In fact

$$g(t) \sim \frac{\overline{\beta}\Gamma(1+1/\beta)}{\pi} t\psi^{-1}(1/t) \quad \text{as } t \to \infty \tag{1.20}$$

THEOREM 1.4. – *Let* X *be a recurrent symmetric Lévy process which satisfies* (1.9) *and for which the function g defined in* (1.11) *is regularly varying at infinity with index* $0 < 1/\overline{\beta} \leqq 1/2$ *or, equivalently, for which* ψ *is regularly varying at zero with index* $1 < \beta \leqq 2$. *Let* $\{L_t^y, (t, y) \in \mathrm{R}^+ \times \mathrm{R}\}$ *denote the local times of* X. *Then*

$$\overline{\lim_{t\to\infty}} \frac{L_t^0}{g(t/\log\log g(t)) \log\log g(t)} = \Gamma(1+1/\overline{\beta})\, \beta^{1/\beta}\, \overline{\beta}^{1/\overline{\beta}} \quad a.s. \tag{1.21}$$

and for any $x \in \mathrm{R}$

$$\overline{\lim_{t\to\infty}} \frac{L_t^0 - L_t^x}{g^{1/2}(t/\log\log g(t)) \log\log g(t)} = \sqrt{2}\,\sigma(x)\,(\Gamma(1+1/\overline{\beta}))^{1/2}\, \alpha^{1/\alpha}\, \overline{\alpha}^{1/\overline{\alpha}} \quad a.s. \tag{1.22}$$

where $\overline{\alpha} = 2\overline{\beta}$ *and* $1/\alpha + 1/\overline{\alpha} = 1$. (1.22) *also holds with the numerator on the left-hand side replaced by* $\sup_{0\leqq u\leqq t} L_u^0 - L_u^x$.

Furthermore, analogous results also hold for symmetric random walks with the obvious modifications.

Writing

$$\beta^{1/\beta}\, \overline{\beta}^{1/\overline{\beta}} = \exp(-(1/\beta)\log 1/\beta - (1/\overline{\beta})\log 1/\overline{\beta}) \tag{1.23}$$

we see that as $\overline{\beta}$ goes to infinity or equivalently as β goes to one the terms in (1.23) go to one. Thus the constants in Theorems 1.1-1.3, in which g

is regularly varying with index zero, are consistent with the constants in (1.21) and (1.22).

Theorem 1.4 is actually a slight improvement over the corresponding results in [MR1]. We will explain this in Section 2.

In Theorem 1.4 it is not necessary to give conditions on p_t, only on $g(t)$. We suspect that Theorem 1.4 also holds when g is slowly varying at infinity. This is certainaly suggested by Theorems 1.2 and 1.3 but there are some gaps.

Remark 1.5. – Both (1.21) and (1.22) can be writen in a simple form involving the L^1 and L^2 norms of the local times. Note that

$$r(t) \equiv E^0(L_t^0) = g(t) \tag{1.24}$$

and

$$s_x(t) \equiv (E^0(L_t^0 - L_t^x))^{1/2} \sim \sqrt{2}\,\sigma(x)\,g^{1/2}(t) \quad \text{as } t \to \infty \tag{1.25}$$

Thus we can write (1.17), (1.18), (1.21) and (1.22) respectively as follows:

$$\overline{\lim_{t\to\infty}} \frac{L_t^0}{r(t/\log\log r(t)) \log\log r(t/\log\log)r(t)}$$

$$= \Gamma(1+1/\overline{\beta})\,\beta^{1/\beta}\,\overline{\beta}^{1/\overline{\beta}} \quad \text{a.s.} \tag{1.26}$$

and

$$\overline{\lim_{t\to\infty}} \frac{L_t^0 - L_t^x}{s_x(t/\log\log s_x(t)) \log\log s_x(t/\log\log)s_x(t)}$$

$$= (\Gamma(1+1/\overline{\beta}))^{1/2}\,\alpha^{1/\alpha}\,\overline{\alpha}^{1/\overline{\alpha}} \quad \text{a.s.} \tag{1.27}$$

We see that for the law of the iterated logarithm the positive term L_t^0 is normalized by it's L^1 norm whereas the essentially symmetric term $L_t^0 - L_t^x$ is normalized by it's L^2 norm. Evaluating the norms at t divided by an iterated logarithm term is a phenomenon that has been observed by other authors.

Some parts of the Theorems hold under weaker hypotheses than stated. We leave it to the interested reader to ferret this out.

The proofs of the Theorems and a further explanation of the Remark are given in Section 2. In Section 3 we show that condition (1.15) can be realized and explain the significance of (1.16). We also point out that for processes on one dimensional spaces, $p_t(0)$ can be taken to be asymptotic to any regularly varying function of index minus one for which (1.12) holds whereas for symmetric random walks on Z^2, $g(n) \leqq C \log n$ for n sufficiently large.

2. PROOFS

We first list some simple observations which we will use in this section. The first Proposition is just a statement of Theorem 1.7.1 [BGT].

PROPOSITION 2.1. – *Let $g(t)$, which is defined in* (1.11), *be regularly varying at infinity with index $0 \leqq \rho$ and set*

$$\kappa(\alpha) = \int_0^\infty e^{-\alpha t} p_t(0)\, dt. \tag{2.1}$$

Then

$$\kappa(\alpha) \sim \Gamma(1+\rho)\, g(1/\alpha) \text{ as } \alpha \to 0 \tag{2.2}$$

where we write $f(x) \sim g(x)$ as $x \to 0$ to signify $\lim_{x\to 0} f(x)/g(x) = 1$ and similarly when $x \to \infty$. Furthermore, a similar statement holds for symmetric random walks with

$$\kappa(\alpha) = \sum_0^\infty e^{-\alpha n} p_n(0) \tag{2.3}$$

PROPOSITION 2.2. – *Let* X *be a Lévy process with local time L_t^0 and define*

$$a(v, u) = \frac{\int_0^u p_s(v)\, ds}{g(u)}. \tag{2.4}$$

Then for all integers $n \geqq 0$ and $t \geqq 0$ and vectors v

$$a(v, t/n)\, n!\, g^n(t/n) \leqq E^v (L_t^0)^n \leqq E^0 (L_t^0)^n \leqq n!\, g^n(t). \tag{2.5}$$

Proof. – To get (2.5) note that

$$\begin{aligned} E^v (L_t^0)^n &= n! \int_0^t p_\cdot(v) \star p_\cdot(0)^{*(n-1)}(u)\, du \\ &= n! \int_{\sum_{i=1}^n s_i \leqq t,\, s_i \geqq 0} \cdots \int p_{s_n}(v) \\ &\quad \times \prod_{i=1}^{n-1} p_{s_i}(0)\, d(s_1) \cdots d(s_n) \end{aligned} \tag{2.6}$$

where $p.(0)^{*n}$ denote the n-fold convolution of $p.(0)$. *See* e.g. (2.21), [MR2]. Using this we see that (2.5) follows easily from (2.6). Also, let us note that by the strong Markov property

$$E^{v}(\phi(L_t^0)) \leqq E^0(\phi(L_t^0)) \tag{2.7}$$

for every increasing function ϕ

For simplicity we introduce the following notation

$$l_2\, g(t) = \log \log g(t) \tag{2.8}$$

$$g_1(t) = g(t/\log \log g(t)). \tag{2.9}$$

PROPOSITION 2.3. – *Let g be slowly varying at infinity. Then for all $a > 0$ there exists a t_a such that for all $t \geqq t_a$*

$$\frac{g(t)}{g_1(t)} \leqq (l_2\, g(t))^a. \tag{2.10}$$

In particular

$$\lim_{t\to\infty} \frac{l_2\, g(t)}{\log \log g_1(t)} = 1. \tag{2.11}$$

Proof. – (2.10) follows easily from the standard representation of slowly varying functions since we can write

$$\frac{g(t)}{g_1(t)} = \exp\left(\int_{t/l_2\, g(t)}^{t} \frac{\varepsilon(u)}{u}\, du\right)$$

where $\lim_{u\to\infty} \varepsilon(u) = 0$. (*See* e.g. Theorem 1.3.1 [BGT]). (2.11) follows immediately from (2.10).

PROPOSITION 2.4. – *Let* X *be a Lévy process which satisfies* (1.9) *and let* L_t^0 *denote the local time of* X. *Assume that g, defined in* (1.11), *is slowly varying at infinity. Then for all $\eta > 0$ sufficiently small there exists a y' such that for all for $y \geqq y'$*

$$E^0\, e^{(1-\eta)\, L_t^0/g(y)} \leqq C_\eta\, e^{t/y} \left(1 + \frac{\displaystyle\int_1^y \frac{g(u)}{u}\, du}{g(y)}\right) \tag{2.12}$$

for all $t \geqq 0$, where C_η is a constant depending only on η.

Proof. – We use the methods of [MR1] to estimate the moment generating function on the left-hand side of (2.12). Let κ be as given in (2.1). We define it's analytic extension to the complex plane by

$$\kappa(z) = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\psi(\lambda) + z}.$$

Let $0 < x_0 < 1$ and set $s\kappa(x_0) = 1$, $s' = (1-\eta)\, s$ and $\Gamma = \{x_0 + iy,\ x_0 > 0\}$. It follows from (2.23) [MR1] that

$$\begin{aligned} E^0\, e^{s'\, L_t^0} - 1 &\leqq \left| \frac{1}{2\pi i} \int_\Gamma e^{zt}\, \frac{s'\,\kappa(z)}{z\,(1 - s'\,\kappa(z))}\, dz \right| \\ &\leqq \frac{s'\, e^{x_0\, t}}{\pi} \int_0^\infty \frac{|\,\kappa(x_0+iy)\,|\, dy}{|\,1 - s'\,\kappa(x_0+iy)\,|\,(x_0^2+y^2)^{1/2}}. \end{aligned}$$

It is easy to see that $|\,\kappa(x_0+iy)\,| \leqq \kappa(x_0)$ and, for $y > 0$, that $\kappa(x_0+y) \leqq \kappa(y)$. Since $s'\,\kappa(x_0) = 1-\eta$, using (2.15) [MR1], we see that this last term

$$\begin{aligned} &\leqq C_\eta\, s e^{x_0\, t} \int_0^\infty \frac{1}{(x_0^2+y^2)^{1/2}}\, \kappa(x_0+y)\, dy \\ &\leqq C_\eta\, s e^{x_0\, t} \left(\kappa(x_0) + \int_{x_0}^\infty \frac{\kappa(y)}{y}\, dy \right) \\ &\leqq C_\eta\, e^{x_0\, t} \left(1 + s \int_{x_0}^\infty \frac{\kappa(y)}{y}\, dy \right). \end{aligned}$$

Note that

$$\int_{x_0}^\infty \frac{\kappa(u)}{u}\, du = \int_1^\infty \frac{\kappa(u)}{u}\, du + \int_1^{1/x_0} \frac{\kappa(1/u)}{u}\, du. \tag{2.13}$$

The first term to the right of the integral sign in (2.13) is finite by (2.8) [MR1] (which uses hypothesis (1.9)). Thus taking $y = 1/x_0$, for x_0 sufficiently small, we see that (2.12) follows from these relationships and (2.2).

We now give *a* a simple upper bound for the distribution of the local times of Lévy processes which is valid for all Lévy process but which is sharp only when *g* satisfies (1.14).

LEMMA 2.5. – *Let* X *be a Lévy process with local time* L_t^0 *and let g be as defined in* (1.11). *Then for all* $x \geqq 0$ *and* $t \geqq 0$

$$P^0 \left(\frac{L_t^0}{g(t)} \geqq x \right) \leqq C\, \sqrt{x+1}\, e^{-x}. \tag{2.14}$$

Proof. – By (2.5) and Chebyshev's inequality we have

$$P^0 \left(\frac{L_t^0}{g(t)} \geqq x \right) \leqq \frac{n!}{x^n} = \frac{n^n\, e^{-n}\, \sqrt{n}}{x^n}\, (1 + O(1/n))$$

independant of *t*. Taking $n = [x]$, we obtain (2.14).

The next bound for the distribution of the local times of Lévy processes is more sophisticated. It depends on Proposition 2.4 but it is useful to us only when g satisfies (1.15).

LEMMA 2.6. – *Let* X *be a Lévy process which satisfies* (1.9) *and let* L_t^0 *denote the local time of* X. *Assume that g is slowly varying at infinity. Then for all* $\varepsilon > 0$ *and* $t = t(\varepsilon)$ *sufficiently large*

$$\mathrm{P}^0\left(\frac{\mathrm{L}_t^0}{g_1(t)} \geqq (1+2\varepsilon)\, l_2\, g(t)\right) \leqq \mathrm{C}_\varepsilon \left(1 + \frac{g(t)}{g_1(t)} \frac{\int_1^t \frac{g(y)}{y}\, dy}{g(t)}\right) e^{-(1+\varepsilon)\, l_2\, g(t)} \tag{2.15}$$

where C_ε *is a constant depending only on* ε.

Proof. – Let

$$y = f(t) \equiv \frac{2t}{\varepsilon l_2\, g(t)}$$

in (2.12). Since g is slowly varying at infinity $g(f(t)) \sim g_1(t)$ as t goes to infinity. Therefore, by (2.12), for all $\delta > 0$ we have that for all t sufficiently large

$$\mathrm{E}^0\, e^{(1-2\delta)\,\mathrm{L}_t^0/g_1(t)} \leqq \mathrm{C}_\delta\, e^{(\varepsilon/2)\, l_2\, g(t)} \left(1 + \frac{g(t)}{g_1(t)} \frac{\int_1^t \frac{g(u)}{u}\, du}{g(t)}\right) \tag{2.16}$$

where C_δ is a constant depending only on δ. (2.15) follows easily by Chebyshev's inequality.

We next obtain a lower bound for the probability distribution in (2.15) that will be useful to us when g satisfies (1.14).

LEMMA 2.7. – *Let* X *be a Lévy process and let* L_t^0 *denote the local time of* X. *Assume that g is slowly varying at infinity. Let x (t) be an increasing function of t such that*

$$\lim_{t\to\infty} \frac{g(t/x(t))}{g(t)} = 1. \tag{2.17}$$

Then for all $\eta > 0$ *there exists an* x_0 *and a* $t' = t'(\eta, x_0, x(\cdot))$ *such that for all* $t \geqq t'$ *and* $x_0 \leqq x \leqq x(t)$

$$\mathrm{P}^0 \left(\frac{\mathrm{L}_t^0}{g(t)} \geqq (1-\eta)\, x \right) \geqq \mathrm{C}_\eta\, e^{-x} \tag{2.18}$$

where C_η *is a constant depending only on* η *and*

$$\mathrm{P}^v \left(\frac{\mathrm{L}_t^0}{g(t)} \geqq (1-\eta)\, x \right) \geqq \mathrm{C}_\eta \left(a(v,\, 2t/(3x))\, e^{-x} - e^{-(1+\eta')\, x} \right) \tag{2.19}$$

for some $\eta' > 0$. [$a(v,\, t/x)$ *is defined in* (2.4).]

Proof. – We first prove (2.18). Let $\mathrm{Y}_t = \mathrm{L}_t^0 / g(t)$ and assume that $x \geqq x_0$ for some x_0 sufficiently large. We note that for any $y > x$ we have, for any integer *n*, that

$$\begin{aligned}
\mathrm{P}^0(\mathrm{Y}_t \geqq x) &\geqq \frac{1}{y^n} \int_x^y u^n\, d\mathrm{P}^0(\mathrm{Y}_t \leqq u) \\
&= \frac{1}{y^n} \mathrm{E}^0(\mathrm{Y}_t^n) - \frac{1}{y^n} \int_0^x u^n\, d\mathrm{P}^0(\mathrm{Y}_t \leqq u) \\
&\quad - \frac{1}{y^n} \int_y^\infty u^n\, d\mathrm{P}^0(\mathrm{Y}_t \leqq u) \\
&\equiv \mathrm{A}_1 - \mathrm{B}_1 - \mathrm{B}_2.
\end{aligned} \tag{2.20}$$

Integrating by parts and using (2.14) we have

$$\begin{aligned}
\mathrm{B}_1 &\leqq \frac{n}{y^n} \int_0^x u^{n-1}\, \mathrm{P}^0(\mathrm{Y}_t \geqq u)\, du \\
&\leqq \frac{\mathrm{C} n x}{y^n} \sup_{0 \leqq u \leqq x} u^{n-1/2}\, e^{-u}.
\end{aligned} \tag{2.20 a}$$

We note that $u^{n-1/2}\, e^{-u}$ is increasing for $u \leqq n - 1/2$. Therefore taking $n = [(1+\delta)\, x]$, for $\delta > 0$, we have that $u^{n-1/2}\, e^{-u}$ is increasing on $0 \leqq u \leqq x$. Therefore

$$\mathrm{B}_1 \leqq \mathrm{C} x^{3/2} \frac{x^n}{y^n}\, e^{-x}.$$

Taking $y = (1+2\delta)\, x$ we see that

$$\begin{aligned}
\mathrm{B}_1 &\leqq \mathrm{C} x^{3/2} (1+2\delta)^{-(1+\delta)\, x}\, e^{-x} \\
&= \mathrm{C} x^{3/2}\, e^{-(1+\delta)\,(2\delta - 2\delta^2 + O(\delta^3))\, x}\, e^{-x} \\
&= \mathrm{C} x^{3/2}\, e^{-(1+2\delta+O(\delta^3))\, x}.
\end{aligned} \tag{2.21}$$

We use the right-hand side of (2.5) to obtain an upper bound for B_2. We have

$$\begin{aligned} B_2 &\leqq \frac{1}{y^{n+[\delta n]}} \int_y^\infty u^{n+[\delta n]}\, dP^0 (Y_t \leqq u) \\ &\leqq \frac{1}{y^{n+[\delta n]}} E(Y_t^{n+[\delta n]}) \\ &\leqq \frac{(n+[\delta n])!}{y^{n+[\delta n]}} \\ &\leqq Cx^2 \frac{[(1+2\delta)\,x]!}{((1+2\delta)\,x)^{(1+2\delta)\,x}} \\ &\leqq Cx^3\, e^{-(1+2\delta)\,x}. \end{aligned} \tag{2.22}$$

Using the left-hand side of (2.5) we see that

$$A_1 \geqq \frac{n!}{y^n} \left(\frac{g(t/n)}{g(t)} \right)^n. \tag{2.23}$$

Note that

$$\begin{aligned} \frac{n!}{y^n} &\geqq Cx^{-1/2} \left(\frac{(1+\delta)\,x}{(1+2\delta)\,x} \right)^{(1+\delta)\,x} e^{-(1+\delta)\,x} \\ &= Cx^{-1/2}\, e^{-(1+\delta)\,x\,(1+\log(1+2\delta)-\log(1+\delta))} \\ &= Cx^{-1/2}\, e^{-(1+2\delta-\delta^2/2+O(\delta^3))\,x}. \end{aligned} \tag{2.24}$$

Using (2.23) and (2.24) and the fact that g is slowly varying at infinity we see that for all x sufficiently large

$$A_1 \geqq Cx^{-1/2}\, e^{-(1+2\delta-\delta^2/2+O(\delta^3))\,x} \left(\frac{g(t/x)}{g(t)} \right)^{[(1+\delta)\,x]}. \tag{2.25}$$

Note that since g is increasing $g(t/x) \geqq g(t/x(t))$ for $x \leqq x(t)$ and so, by (2.17)

$$\frac{g(t/x)}{g(t)} \geqq e^{-\delta^3} \tag{2.26}$$

for all $x \leqq x(t)$ for all t sufficiently large. Combining all these inequalities completes the proof of (2.18).

The proof of (2.19) follows exactly as the proof of (2.18) once we make a few observations. Instead of (2.20) we write

$$\begin{aligned} \mathrm{P}^v (\mathrm{Y}_t \geqq x) &= \frac{1}{y^n} \mathrm{E}^v (\mathrm{Y}_t^n) - \frac{1}{y^n} \int_0^x u^n \, d\mathrm{P}^v (\mathrm{Y}_t \leqq u) \\ &\quad - \frac{1}{y^n} \int_y^\infty u^n \, d\mathrm{P}^v (\mathrm{Y}_t \leqq u) \\ &\equiv \mathrm{A}_1' - \mathrm{B}_1' - \mathrm{B}_2'. \end{aligned} \tag{2.27}$$

It follows from (2.5) and (2.23) that

$$\mathrm{A}_1' \geqq a(v, t/n) \frac{n!}{y^n} \left(\frac{g(t/n)}{g(t)} \right)^n \geqq a(v, 2t/(3x)) \frac{n!}{y^n} \left(\frac{g(t/n)}{g(t)} \right)^n.$$

Also by (2.7)

$$\mathrm{E}^v (\phi(\mathrm{Y}_t^n)) \leqq \mathrm{E}^0 (\phi(\mathrm{Y}_t^n)). \tag{2.28}$$

This shows that B_2' can be bounded above by the upper bound given for B_2 in (2.22). It also shows that B_1' is less than or equal to the last term in (2.21). The reason for this is that the last term in (2.21) was obtained using Chebyshev's inequality to find an upper bound for $\mathrm{P}^0(\mathrm{Y}_t \geqq u)$. By (2.28) the same technique gives an estimate no larger for $\mathrm{P}^v(\mathrm{Y}_t \geqq u)$. Using these bounds for A_1', B_1' and B_2' we get (2.19).

We now obtain a lower bound for the probability distribution in (2.15) that will be useful to us when g satisfies (1.15).

LEMMA 2.8. – *Let* X *be a Lévy process and let* L_t^0 *denote the local time of* X. *Assume that g is slowly varying at infinity. Let* $x(t)$ *be an increasing function of t such that* $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} t/x(t) = \infty$ *and*

$$\frac{\displaystyle\int_1^t \frac{g(y)}{y} \, dy}{g(t)} = o(e^{\varepsilon x(t)}), \qquad \forall \varepsilon > 0. \tag{2.29}$$

Then for all $\eta > 0$ *there exists a* $t' = t'(\eta, x(\cdot))$ *such that for all* $t \geqq t'$

$$\mathrm{P}^0 \left(\frac{\mathrm{L}_t^0}{g(t/x(t))} \geqq (1-\eta) x(t) \right) \geqq \mathrm{C}_\eta \, e^{-x(t)} \tag{2.30}$$

where C_η *is a constant depending only on* η *and*

$$\begin{aligned} &\mathrm{P}^v \left(\frac{\mathrm{L}_t^0}{g(t/x(t))} \geqq (1-\eta) x(t) \right) \\ &\quad \geqq \mathrm{C}_\eta \left(a(v, 2t/(3x(t))) \, e^{-x(t)} - e^{-(1+\eta') x(t)} \right) \end{aligned} \tag{2.31}$$

for some $\eta' > 0$.

Proof. – We first prove (2.30). The proof is the essentially the same as that of Lemma 2.7 except that the inequalities are derived differently. Let $Y_t = (1-2\eta)\, L_t^0 / g(t/n)$ and consider the terms A_1, B_1 and B_2 in (2.20) for this Y_t. Denote them by A_1', B_1' and B_2'. In order to obtain an upper bound for B_1' we use (2.12) with $y = t/(\eta n)$, where η is such that $\eta n \geqq 1$, and Chebysev's inequality, to get

$$P^0\left(\frac{(1-\eta)\, L_t^0}{g(t/(\eta n))} \geqq u\right)$$
$$\leqq C_\eta\, e^{\eta n}\left(1 + \frac{g(t)}{g(t/(\eta n))}\,\frac{\int_1^t \frac{g(y)}{y}\, dy}{g(t)}\right) e^{-u}. \tag{2.32}$$

As in the proof of Lemma 2.7 we will set $n = [(1+\delta)\, x(t)]$. The condition that $t/x(t)$ is sufficiently large, together with the slow variation of g at infinity and the proof of Proposition 2.3 enables us to write (2.32) as

$$P^0\left(\frac{(1-2\eta)\, L_t^0}{g(t/(\eta n))} \geqq u\right) \leqq e^{\eta n}\, h(n)\, e^{-u}$$

where $h(n) = o(e^{\varepsilon n})$ for all $\varepsilon > 0$. Using this inequality in (2.20 *a*) we see that B_1' is bounded above by the last term in (2.21) multiplied by $e^{\eta n}\, h(n)$. Continuing we have by (2.5) that

$$A_1' = \frac{1}{y^n}\, E^0(Y_t^n) \geqq \frac{(1-2\eta)^n\, n!}{y^n}. \tag{2.33}$$

It also follows from (2.12) that

$$E^0\left(\frac{(1-\eta)\, L_t^0}{g(t/(\eta n))}\right)^m \leqq m!\, h(n)\, e^{-\eta n}$$

so, again by the slow variation of g, we get

$$E^0\, Y_t^{[n+[\delta n]]} \leqq C\,[n+[\delta n]]!\, h(n)\, e^{\eta n}.$$

Therefore we see that B_2' is bounded above by the last term in (2.22) multiplied by $e^{\eta n}\, h(n)$. Thus we get the same critical expressions as in the proof of Lemma 2.7 except for the terms $(1-2\eta)^n, h(n)$ and $e^{\eta n}$. But we can take $h(n) \leqq e^{\eta n}$ and taking $\eta = \delta^3$ we see that these terms don't effect the arguments used to prove Lemma 2.7. Thus we get the proof of (2.30). The extension of proof of (2.30) to the proof of (2.31) is exactly the same as the analogous extension in the previous Lemma.

We now develop some technical lemmas which will be used in the proofs of Theorems 1.2 and 1.3. We can write

$$g(t) = \exp\left(\int_1^t \frac{\varepsilon(u)}{u}\,du\right)$$

where

$$\varepsilon(u) = \frac{ug'(u)}{g(u)}. \tag{2.34}$$

That is, the general expression for slowly varying functions has this simplified form when g is differentiable and, of course, $g'(t) = p_t(0)$. Recall that

$$\lim_{u \to \infty} \varepsilon(u) = 0 \tag{2.35}$$

and note that by (1.10), $\varepsilon(u)$ is slowly varying at infinity. For $\theta > 1$ we define the sequence $\{t_n\}_{n=1}^{\infty}$ by

$$g(t_n) = \theta^n. \tag{2.35 a}$$

This is well defined since g is strictly increasing. We set

$$\Delta_n = \frac{t_n - t_{n-1}}{\log n} \qquad \text{and} \qquad \gamma_n = \frac{\Delta_n}{t_{n-1}}$$

and note the following simple facts:

PROPOSITION 2.9.

$$\lim_{t \to \infty} \frac{t_n}{t_{n-1}} = \infty \tag{2.36}$$

$$\lim_{t \to \infty} \frac{\log t_n}{n} = \infty \tag{2.37}$$

and

$$\frac{1}{g(\Delta_n)} \leqq \frac{\log n}{\theta^n} \quad \textit{for all } n \textit{ sufficiently large.} \tag{2.38}$$

Proof. – Since

$$\theta = \frac{g(t_n)}{g(t_{n-1})} = \exp\left(\int_{t_{n-1}}^{t_n} \frac{\varepsilon(u)}{u}\,du\right) = \exp\left(\varepsilon(s)\log\frac{t_n}{t_{n-1}}\right)$$

for some $t_{n-1} \leqq s \leqq t_n$, we get (2.36) by (2.35). To get (2.37) we note that since $g(t)$ is slowly varying $\theta^n = g(t_n) < t_n^{\eta}$, for all $\eta > 0$ and $n \geqq n(\eta)$

for some $n(\eta)$ sufficiently large. This implies (2.37). To get (2.38) we note that by (2.36)

$$\Delta_n \geqq \frac{t_n}{2 \log n} \quad \text{for all } n \text{ sufficiently large.}$$

Therefore

$$\frac{1}{g(\Delta_n)} \leqq \frac{1}{g(t_n/2 \log n)} \quad \text{for all } n \text{ sufficiently large.} \tag{2.39}$$

Also note that

$$\frac{g(t_n/2 \log n)}{g(t_n)} = \exp\left(-\int_{t_n/(2 \log n)}^{t_n} \frac{\varepsilon(u)}{u}\, du\right)$$

$$= \exp(-\varepsilon(s_n) \log(2 \log n)) \tag{2.40}$$

for some $t_n/(2 \log n) \leqq s_n \leqq t_n$. We see from (2.37) that $\lim_{n\to\infty} s_n = \infty$ and hence, by (2.35) and (2.40)

$$g\left(\frac{t_n}{2 \log n}\right) > \frac{g(t_n)}{\log n} \quad \text{for all } n \text{ sufficiently large.} \tag{2.41}$$

Using (2.41) in (2.39) we get (2.38).

Let $\beta_n = t_{n-1} + \Delta_n/2$ and note that $\beta_n < t_n$. Also let

$$\mathrm{U}_n = \frac{g(\beta_n) - g(t_{n-1})}{g(t_{n-1})} \qquad \text{and} \qquad \mathrm{V}_n = \log\left(\frac{g(t_n)}{g(\beta_n)}\right). \tag{2.42}$$

PROPOSITION 2.10. – *There exists a constant* C > 0 *such that for all n sufficiently large*

$$\varepsilon(t_n) \geqq \frac{\mathrm{C}}{(\log n)} \mathrm{V}_n \tag{2.43}$$

$$\mathrm{U}_n \geqq \frac{\mathrm{C}}{(\log n)} \varepsilon(t_{n-1}). \tag{2.44}$$

Proof. – By integration by parts and the fact that $\varepsilon(u)$ is slowly varying at infinity, we have

$$\mathrm{V}_n = \int_{\beta_n}^{t_n} \frac{\varepsilon(u)}{u}\, du = \int_{\beta_n}^{t_n} \frac{1}{u}\, d\left(\int_{\beta_n}^{u} \varepsilon(s)\, ds\right)$$

$$= \frac{1}{t_n} \int_{\beta_n}^{t_n} \varepsilon(s)\, ds + \int_{\beta_n}^{t_n} \frac{1}{u^2} \left(\int_{\beta_n}^{u} \varepsilon(s)\, ds\right) du$$

By (2.36) and (2.37) $\lim_{n\to\infty} \beta_n = \infty$. Thus

$$\begin{aligned} V_n &\leqq \frac{1}{t_n} \int_1^{t_n} \varepsilon(s)\, ds + \int_{\beta_n}^{t_n} \frac{1}{u^2} \left(\int_1^{t_n} \varepsilon(s)\, ds \right) du \\ &\leqq (1 + \delta(n)) \left(\varepsilon(t_n) + t_n\, \varepsilon(t_n) \int_{\beta_n}^{t_n} \frac{1}{u^2}\, du \right) \\ &\leqq (1 + \delta(n)) \left(\varepsilon(t_n) + \frac{t_n\, \varepsilon(t_n)}{\beta_n} \right) \end{aligned} \tag{2.45}$$

where $\lim_{n\to\infty} \delta(n) = 0$ and where, at the next to last step, we use the fact that $\varepsilon(u)$ is slowly varying at infinity. Using (2.36) in (2.45) we get (2.43). To obtain (2.44) we note that

$$U_n = \exp\left(\int_{t_{n-1}}^{\beta_n} \frac{\varepsilon(u)}{u}\, du \right) - 1 \geqq \int_{t_{n-1}}^{\beta_n} \frac{\varepsilon(u)}{u}\, du$$

Therefore if $\beta_n \geqq 2t_{n-1}$

$$U_n \geqq \int_{t_{n-1}}^{2t_{n-1}} \frac{\varepsilon(u)}{u}\, du \geqq (1 - \delta(n))\, \varepsilon(t_{n-1}) \log 2$$

where $\lim_{n\to\infty} \delta(n) = 0$ and we use the fact that $\varepsilon(u)$ is slowly varying at infinity. Thus (2.44) is satisfied when $\beta_n \geqq 2t_{n-1}$. Let us now assume that $\beta_n < 2t_{n-1}$. Then, using the slow variation of $\varepsilon(u)$ again, we get

$$\begin{aligned} U_n &= \exp\left(\int_{t_{n-1}}^{t_{n-1}(1+\gamma_n/2)} \frac{\varepsilon(u)}{u}\, du \right) - 1 \\ &\geqq \exp\left((1 - \delta(n))\, \varepsilon(t_{n-1}) \log(1 + \gamma_n/2) \right) - 1 \\ &\geqq C\, \varepsilon(t_{n-1})\, \gamma_n \geqq \frac{C\, \varepsilon(t_{n-1})}{\log n} \end{aligned}$$

where we use (2.36) and the fact that $\gamma_n < 1$. Thus we get (2.44) in this case also.

The result which we will use in the proofs of Theorems 1.2 and 1.3 is given in the following Lemma.

LEMMA 2.11. – *There exists a constant* $C > 0$ *such that for all n sufficiently large*

$$U_{n-1} + U_n \geqq \frac{C}{(\log n)^2} \tag{2.46}$$

Proof. – By the definition of the sequence $\{ g(t_n) \}$, either

$$\frac{g(\beta_{n-1})}{g(t_{n-2})} \geqq \theta^{1/2} \qquad \text{or} \qquad V_{n-1} \geqq \frac{1}{2} \log \theta \tag{2.47}$$

If the first statement in (2.47) holds then $U_{n-1} \geqq \theta^{1/2} - 1$. If the second statement in (2.47) holds then by (2.43) and (2.44), U_n is greater than or equal to the right hand side of (2.46), for an appropriate choice of the constant C. Thus we have (2.46).

Proof of Theorem 1.2, (1.17). – We first obtain the upper bound in (1.17) when g satisfies (1.14). Recall the terms defined in (2.8), (2.9) and (2.35 *a*). By Lemma 2.5 we have that for all integers $n \geqq 2$ and all $\varepsilon > 0$

$$P^0 \left(\frac{L^0_{t_n}}{g(t_n)\, l_2\, g(t_n)} \geqq (1+\varepsilon) \right) \leqq C \sqrt{\log n}\, e^{-(1+\varepsilon) \log n}. \tag{2.48}$$

Therefore, by the Borel-Cantelli lemma

$$\overline{\lim_{n\to\infty}} \frac{L^0_{t_n}}{g(t_n)\, l_2\, g(t_n)} \leqq 1+\varepsilon \quad \text{a.s.} \tag{2.49}$$

Next suppose that g satisfies (1.15). Let $\{ u_n \}_{n=1}^{\infty}$ be such that $g_1(u_n) = \theta^n$, (recall $\theta > 1$), and note that by (2.10) and (2.11)

$$\frac{g(u_n)}{g_1(u_n)} \leqq \log n. \tag{2.50}$$

Also, since the following integral diverges by (1.12),

$$\int_1^{u_n} \frac{g(y)}{y}\, dy \leqq C \sum_{k=n_0}^{n} g(u_k) \log \frac{u_k}{u_{k-1}} \tag{2.51}$$

for all $n \geqq n_0$, for some n_0 sufficiently large. Since g satisfies (1.15) there exists a $\delta > 0$ such that

$$\frac{g(u_n/\log g(u_n))}{g(u_n)} < 1 - \delta$$

for all n sufficiently large. Choose $\theta \leqq \theta_0$, such that $(1-\delta) < 1/\theta$. This implies that $u_n/\log g(u_n) < u_{n-1}$, or equivalently, using (2.10) and (2.11), that

$$\log \frac{u_n}{u_{n-1}} \leqq (1+\delta') \log n \tag{2.52}$$

for some $\delta' > 0$ and all n sufficiently large. Using (2.50)-(2.52) we now see that

$$\int_1^{u_n} \frac{g(y)}{y}\, dy \leqq \mathrm{C} \sum_{k=n_0}^{n} \theta^k (\log\, k)^2$$
$$\leqq \mathrm{C} g_1(u_n)(\log\, n)^2 \tag{2.53}$$

for all n sufficiently large. Using (2.53) in (2.15) we get

$$\mathrm{P}^0\left(\frac{\mathrm{L}^0_{u_n}}{g_1(u_n)\, l_2\, g(u_n)} \geqq (1+2\varepsilon)\right) \leqq \mathrm{C}(\log\, n)^2\, e^{-(1+\varepsilon/2)\log\, n}. \tag{2.54}$$

Thus when g satisfies (1.15) we have that

$$\overline{\lim_{n\to\infty}} \frac{\mathrm{L}^0_{u_n}}{g_1(u_n)\, l_2\, g(u_n)} \leqq (1+2\varepsilon) \quad \text{a.s.} \tag{2.55}$$

Lastly, let us consider the case when g satisfies (1.16). If

$$\frac{g(u_n)}{g(u_n/\log\, n)} > 1+\varepsilon \tag{2.56}$$

for any $\varepsilon > 0$ then, by (2.34) and the first part of the proof of Proposition 2.10

$$\frac{g(u_n)}{u_n\, g'(u_n)} \leqq \mathrm{C}_\varepsilon \log\, n \tag{2.57}$$

for some constant $\mathrm{C}_\varepsilon < \infty$ depending on ε. Thus we can use (2.15), (2.10) and (2.11) to obtain (2.54) when (2.56) holds. On the other hand suppose

$$\frac{g(u_n)}{g(u_n/\log\, n)} < 1+\varepsilon < \theta.$$

Then we can use Lemma 2.5 as we did in obtaining (2.48) along with (2.11) to get

$$\mathrm{P}^0\left(\frac{\mathrm{L}^0_{u_n}}{g_1(u_n)\, l_2\, g(u_n)} \geqq (1+2\varepsilon)\right) \leqq \mathrm{C}\sqrt{\log\, n}\; e^{-(1+\varepsilon)\log\, n}.$$

So we see that (2.54) holds in this case also. Thus we get (2.55) when g satisfies (1.16). By taking θ arbitrarily close to one it is simple to interpolate and thereby pass from (2.49) or (2.55) to the upper bound in (1.17). [Note that since g is slowly varying at infinity $t/\log\,\log\, g(t)$ is regularly varying at infinity with index one and hence is assymptotic to an increasing function.

Consequently, since g is monotonically increasing, g_1 itself is assymptotic to an increasing function. Also we use the fact that g satisfies (1.14) to go from (2.49) to (1.17).]

We now show that for any $\varepsilon > 0$

$$\overline{\lim_{n\to\infty}} \frac{L^0_{t_n}}{g_1(t_n)\, l_2\, g(t_n)} \geqq 1-\varepsilon \quad \text{a.s.} \tag{2.58}$$

for all θ sufficiently large. It is sufficient to show that

$$\overline{\lim_{n\to\infty}} \frac{L^0_{t_n} - L^0_{t_{n-1}}}{g_1(t_n)\, l_2\, g(t_n)} \geqq 1-\varepsilon \quad \text{a.s.} \tag{2.59}$$

Let $s_n = t_n - t_{n-1}$. It follows from (2.36) and the slow variation g that

$$g(t_n) \sim g(s_n) \qquad \text{and} \qquad g_1(t_n) \sim g_1(s_n). \tag{2.60}$$

Using this, Lévy's Borel-Cantelli lemma and the Markov property, (2.59) will follow if we show that

$$\sum_{n=1}^{\infty} P^{X_{t_{n-1}}}\left(\frac{L^0_{s_n}}{g_1(s_n)\, l_2\, g(s_n)} \geqq 1-\varepsilon\right) = \infty \quad \text{a.s.} \tag{2.61}$$

(*See* the proof of Theorem 1.1 in [MR1] for more details).

In Lemmas 2.7 and 2.8 take $t = s_n$ and $x = x(s_n) = (1-\eta'/2)\, l_2\, g(s_n)$. Suppose that g satisfies (1.14), then referring to Lemma 2.7, we see that (2.17) is satisfied. Thus by (2.19) we have that

$$\begin{aligned} P^v &\left(\frac{L^0_{s_n}}{g_1(s_n)\, l_2\, g(s_n)} \geqq (1-\eta-\eta'/2)\right) \\ &\geqq C_\eta \left(a(v, s_n/2 \log n)\, \frac{1}{n^{1-\eta'/2}} - \frac{1}{n^{1+\eta'/4}}\right) \end{aligned} \tag{2.62}$$

where $\eta, \eta' > 0$ are sufficiently small.

To handle the case when g satisfies (1.15) or (1.16) note that if (2.17) or (2.29) hold along a sequence say $\{v_n\}$ going to infinity then the conclusions of Lemmas 2.7 and 2.8 hold along this sequence for n sufficiently large. Now suppose that g satisfies (1.15). It is easy to see that (2.50)-(2.52) also hold with u_{n-1} and u_n replaced by t_{n-1} and t_n. Using these, analagous to (2.53), we get

$$\int_1^{t_n} \frac{g(y)}{y}\, dy \leqq C g(t_n)(\log n) \tag{2.62 a}$$

This implies that (2.29) is satisfied along the sequence $\{t_n\}_{n=n_0}^{\infty}$ for some n_0 sufficiently large, for $x(t_n)$ as defined in the previous paragraph. Thus by Lemma 2.8 for $\{t_n\}_{n=n_0}^{\infty}$ and (2.60) we get (2.62).

Finally suppose that g satisfies (1.16). It is easy to see that if (2.56) holds with u_n replaced by t_n then so does (2.57). In this case we can use Lemma 2.8 for t_n, if n is sufficiently large, and (2.60) to obtain (2.62). If (2.56), with u_n replaced by t_n, does not hold then (2.62) follows from Lemma 2.7 for t_n and (2.60) as long as n is sufficiently large. Thus we also have (2.62) when g satisfies (1.16).

We see from (2.61) and (2.62) that in order to obtain (2.58) it is enough to show that

$$\sum_{n=1}^{\infty} a\left(\mathrm{X}_{t_{n-1}}, s_n/2 \log n\right) \frac{1}{n^{1-\eta'/2}} = \infty \quad \text{a.s.} \tag{2.63}$$

We now develop some material which will be used to obtain (2.63). We note that

$$\begin{aligned} \mathrm{E}^0\left(\int_0^s p_u(\mathrm{X}_t)\, du\right) &= \int_0^s \int_{-\infty}^{\infty} p_u(y)\, p_t(y)\, dy\, du \\ &= \int_0^s p_{u+t}(0)\, du \end{aligned} \tag{2.64}$$

and for $t_1 < t_2$

$$\begin{aligned} &\mathrm{E}^0\left(\int_0^{s_2} p_{u_2}(\mathrm{X}_{t_2})\, du_2 \int_0^{s_1} p_{u_1}(\mathrm{X}_{t_1})\, du_1\right) \\ &= \int_0^{s_2}\int_0^{s_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{u_2}(y)\, p_{u_1}(x)\, p_{t_1}(x)\, p_{t_2-t_1} \\ &\quad \times (y-x)\, dx\, dy\, du_1\, du_2. \end{aligned} \tag{2.65}$$

Integrating on y and using the symmetry of $p.$ and fact that $p_t(x) < p_t(0)$ for all x we see that this last term

$$\begin{aligned} &= \int_0^{s_2}\int_0^{s_1}\int_{-\infty}^{\infty} p_{u_2+t_2-t_1}(x)\, p_{u_1}(x)\, p_{t_1}(x)\, dx\, du_1\, du_2 \\ &< \int_0^{s_1}\int_0^{s_2} p_{u_2+t_2-t_1}(0)\, du_2 \int_{-\infty}^{\infty} p_{u_1}(x)\, p_{t_1}(x)\, dx\, du_1 \\ &= \int_0^{s_2} p_{u_2+t_2-t_1}(0)\, du_2 \int_0^{s_1} p_{u_1+t_1}(0)\, du_1. \end{aligned} \tag{2.66}$$

Combining (2.64)-(2.66) we see that

$$\frac{\mathrm{E}^0\left(\int_0^{s_2} p_{u_2}(\mathrm{X}_{t_2})\,du_2 \int_0^{s_1} p_{u_1}(\mathrm{X}_{t_1})\,du_1\right)}{\mathrm{E}^0\left(\int_0^{s_2} p_{u_2}(\mathrm{X}_{t_2})\,du_2\right)\mathrm{E}^0\left(\int_0^{s_1} p_{u_1}(\mathrm{X}_{t_1})\,du_1\right)} < \frac{\int_0^{s_2} p_{u_2+t_2-t_1}(0)\,du_2}{\int_0^{s_2} p_{u_2+t_2}(0)\,du_2}. \tag{2.67}$$

Now let us assume that $t_2 - t_1 > (1-\varepsilon)\,t_2$ for some ε sufficiently small. Then $u_2 + t_2 - t_1 > (1-\varepsilon)(u_2+t_2)$ and since $p.$ is regularly varying at infinity with index minus one we see, that if t_2 is sufficiently large, then

$$p_{u_2+t_2-t_1}(0) < (1+2\varepsilon)\,p_{u_2+t_2}(0). \tag{2.68}$$

If (2.68) holds then the right-hand side of (2.67) is less that $1+2\varepsilon$. Recalling the definition of $a(\mathrm{X}_{t_{n-1}}, s_n/2 \log n)$ given in (2.4), (2.67) and (2.68) show that

$$\frac{\mathrm{E}^0\left(a(\mathrm{X}_{t_{n-1}}, s_n/2\log n)\,a(\mathrm{X}_{t_{j-1}}, s_j/2\log j)\right)}{\mathrm{E}^0\left(a(\mathrm{X}_{t_{n-1}}, s_n/2\log n)\right)\mathrm{E}^0\left(a(\mathrm{X}_{t_{j-1}}, s_j/2\log j)\right)} \leqq 1+2\varepsilon \tag{2.69}$$

for all $\varepsilon > 0$ if $n > j \geqq \mathrm{N}(\varepsilon)$ for some $\mathrm{N}(\varepsilon)$ sufficiently large. Also, since $g(s_n/2\log n) < g(t_n) = \theta g(t_{n-1})$, it follows from (2.64) that

$$\begin{aligned}\mathrm{E}^0\left(a(\mathrm{X}_{t_{n-1}}, s_n/2\log n)\right) &= \frac{g(t_{n-1} + s_n/2\log n) - g(t_{n-1})}{g(s_n/2\log n)}\\ &\geqq \frac{1}{\theta}\,\mathrm{U}_n\end{aligned} \tag{2.70}$$

where U_n is defined in (2.42). Therefore, it follows from Lemma 2.11, that

$$\mathrm{E}^0\left(\sum_{n=\mathrm{N}(\varepsilon)}^{\infty} a(\mathrm{X}_{t_{n-1}}, s_n/2\log n)\,\frac{1}{n^{1-\eta'/2}}\right) = \infty. \tag{2.71}$$

Also, since $a(\mathrm{X}_{t_{n-1}}, s_n/2\log n) \leqq 1$ we see that

$$\sum_{n=\mathrm{N}(\varepsilon)}^{\infty} \mathrm{E}^0\left(a(\mathrm{X}_{t_{n-1}}, s_n/2\log n)\,\frac{1}{n^{1-\eta'/2}}\right)^2 < \infty. \tag{2.72}$$

(2.69)-(2.72) enable us to use the Paley-Zygmund Lemma (*see* e.g. [Ka] Inequality 2, pg. 8) to verify (2.63). This completes the proof of Theorem 1.2, (1.17).

The next proposition is an analogue of Proposition 2.2 for the differences of local times.

PROPOSITION 2.12. – *Let* X *be a Lévy process with local times* $\{ L_t^x, (t,x) \in R^+ \times R \}$ *and define*

$$Z_t = Z_t(x) = \frac{L_t^0 - L_t^x}{\sqrt{2}\sigma(x)}. \tag{2.73}$$

Then for all integers $n \geqq 1$, *real numbers* v *and* t *sufficiently large*

$$\frac{1}{3} a(v, t/2n)(2n)!\, g^n(t/2n) \leqq E^v(Z_t)^{2n} \leqq (2n)!\, g^n(t) \tag{2.74}$$

Proof. – In the same notation used in (2.6) we see from (2.33), [MR2] that

$$\begin{aligned} &E^v(L_t^0 - L_t^x)^{2n} \\ &\quad = (2n)! \int_0^t \Big((p.(v) + p.(x-v)) \star (p.(0) + p.(x))^{*(n-1)} \\ &\qquad \star (p.(0) - p.(x))^{*n} \Big)(u)\, du < (2n)!\, 2^n \left(\int_0^t p_s(0)\, ds \right)^n \\ &\qquad \times \left(\int_0^t (p_s(0) - p_s(x))\, ds \right)^n < (2n)!\, 2^n g^n(t)\, \sigma^{2n}(x) \end{aligned} \tag{2.75}$$

which gives us the right-hand side of (2.74). Note that by (1.13)

$$\begin{aligned} \int_0^t (p_s(0) + p_s(x))\, ds &= 2 \int_0^t p_s(0)\, ds - \int_0^t (p_s(0) - p_s(x))\, ds \\ &\sim 2g(t) \quad \text{as } t \to \infty. \end{aligned} \tag{2.76}$$

Using this and the first equation in (2.75) we see that

$$\begin{aligned} &E^v(L_t^0 - L_t^x)^{2n} \\ &\geqq (2n)! \frac{\displaystyle\int_0^{t/2n} (p_s(v) + p_s(x-v))\, ds}{\displaystyle\int_0^{t/2n} (p_s(0) + p_s(x))\, ds} \left(\int_0^{t/2n} (p_s(0) + p_s(x))\, ds \right)^n \end{aligned}$$

$$\times \left(\int_0^{t/2n} (p_s(0) - p_s(x))\, ds \right)^n$$
$$> \frac{1}{3} (2n)!\, 2^n\, a(v,\, t/2n)\, g^n(t/2n)\, \sigma^{2n}(x) \tag{2.77}$$

for all t sufficiently large. This gives the left-hand side of (2.74).

The next proposition is an analogue of Proposition 2.4 for the differences of local times.

PROPOSITION 2.13. – *Let* X *be a Lévy process which satisfies* (1.9) *with local times* $\{L_t^x,\ (t,\, x) \in R^+ \times R\}$ *Assume that* g *is slowly varying at infinity and let* Z_t *be as defined in* (2.73). *Let* ε *be a random variable satisfying* $P(\varepsilon = 1) = P(\varepsilon = -1) = 1/2$ *which is independent of* X. *Then for all* $\eta > 0$ *sufficiently small and for all* $t \geqq 0$ *there exists an* y' *such that for all for* $y \geqq y'$

$$E^0\, e^{(1-\eta)\, \varepsilon\, Z_t/g^{1/2}(y)} \leqq C_\eta\, e^{t/y} \left(1 + \frac{\displaystyle\int_1^y \frac{g(u)}{u}\, du}{g(y)} \right) \tag{2.78}$$

where C_η *is a constant depending only on* η.

Proof. – This proof is similar to the proof of Proposition 2.4 except that we use the first equation given in Proof of Theorem 2.2, (2.10) [MR1]. [In adapting the proof of [MR1] we take $s = (2\sigma(x)\, \kappa(x_0)^{-1/2}$, $x_0 = 1/y$ and use Lemma 2.1 of this paper].

Proof of Theorem 1.2, (1.18). – Comparing (2.5) and (2.12) with (2.74) and (2.78) we see that $L_t^0/g(t)$ and $\varepsilon Z_t/g^{1/2}(t)$ satisfy essentially the same moment and moment generating function inequalities. Especially when we note that, by the slow variation of g, $a(v,\, 2u) \sim a(v,\, u)$ as u goes to infinity and that it is alright to restrict ourselves to even moments. Inequalities (2.5) and (2.12) were used to prove (1.17) for $L_t^0/g(t)$ and therefore, except at two points the proofs also hold for $\varepsilon Z_t/g^{1/2}(t)$. [Note that (1.18) remains the same with or without multiplying the numerator on the left-hand side by ε].

The first of these points is minor. We did not provide an analogue of (2.7) for Z_t. However, the only place we use this result in the proof of (1.17) is in (2.28) and we only used it to show that $E^v((L_t^0)^m)$ was bounded above by the upper bound given for $E^0((L_t^0)^m)$ in (2.5), where m is an integer.

This is also true, by (2.74), for εZ_t for even integers. As we remarked above it is alright to restrict ourselves to even integers.

The second point, which is more serious, relates to our use of the monotonicity of L_t^0 in t to interpolate in (2.49) and (2.55). Clearly, $L_t^0 - L_t^x$ is not monotonic in t. We handled this point in [MR1] by a martingale argument. Following the proof of Theorem 1.2 in [MR1] and using the triangle inequality and Lemma 2.9 of [MR1] we get

$$E^0 \exp\left(s \sup_{0 \leqq u \leqq t} (L_u^0 - L_u^x)\right) \leqq 4 e^{s\sigma^2(x)} E^0 \exp\left(s\varepsilon\left(L_t^0 - L_t^x\right)\right)$$

for all $s \geqq 0$. This shows that all the upper bounds for the moments and moment generating function of εZ_t that we obtained also hold for $\varepsilon \sup_{0 \leqq u \leqq t} Z_u$. Thus, analogous to (2.49), we obtain

$$\overline{\lim_{n\to\infty}} \frac{\sup_{0 \leqq t \leqq t_n} (L_t^0 - L_t^x)}{\sqrt{2}\sigma(x)\, g^{1/2}(t_n)\, l_2\, g(t_n)} \leqq (1+\varepsilon) \quad \text{a.s.}$$

when g satisfies (1.14) and the same expression with g replaced by g_1 and t_n replaced by u_n when g satisfies (1.15) or (1.16). Furthermore, the interpolation between the $\{t_n\}$ or $\{u_n\}$ is simple. This completes the proof both of (1.18) and the comment immediately following it.

Proof of Theorem 1.3. – Two sets of inequalities are used in the proof of Theorem 1.2. Those coming from [MR1] and those given in Proposition 2.2 which are taken from [MR2]. In each of these references we explained why these inequalities are also valid for symmetric random walks, (with obvious modifications). Since the proof of Theorem 1.2 only depends on these inequalities the same proof also gives this Theorem.

The next Proposition will be used in the proof of Theorem 1.1. It must be well known. Nevertheless, for completeness and lack of a suitable reference we will include a proof.

PROPOSITION 2.14. – *Let* X *be a symmetric random walk on* Z^2 *and let* p_n *be the transition probabilities. Then*

$$p_n(0) \leqq \frac{C}{n}. \tag{2.79}$$

Proof. – By the assumptions on Y_1 the characteristic function ϕ is not degenerate. This implies that

$$\psi(\lambda) = (1 - \phi(\lambda)) \geqq c(\lambda_1^2 + \lambda_2^2) \qquad \forall 0 \leqq (\lambda_1^2 + \lambda_2^2)^{1/2} \leqq \rho_0 \tag{2.80}$$

for some $0 < \rho_0 < 1/10$, where $\lambda = (\lambda_1, \lambda_2)$. Furthermore, since $\phi(\lambda)$ is continuous, $\psi(\lambda)$ is bounded away from zero on $[-\pi/2, \pi/2]^2 \cap \{ (\lambda_1^2 + \lambda_2^2)^{1/2} \leqq \rho_0 \}^c$. By this observation, (1.4) and (2.80) and the periodicity of $\psi(\lambda)$, we have that

$$\begin{aligned} p_n(0) &\leqq C \left(\int_{(\lambda_1^2+\lambda_2^2)^{1/2} \leqq \rho_0} e^{-n\psi(\lambda)/2} \, d\lambda \right. \\ &\quad \left. + \int_{[-\pi/2, \pi/2]^2 \cap \{ (\lambda_1^2+\lambda_2^2)^{1/2} \leqq \rho_0 \}^c} e^{-n\delta} \, d\lambda \right) \\ &\leqq C \left(\int_0^{\rho_0} e^{-nc\rho^2/2} \rho \, d\rho + e^{-n\delta} \right) \\ &\leqq C \left(\frac{1}{n} \left(1 - e^{-nc\rho_0^2/2} \right) + e^{-n\delta} \right) \end{aligned} \tag{2.81}$$

which gives us (2.79).

Proof of Theorem 1.1. – The main point here is that (1.14) is satisfied by symmetric random walks on Z^2. In showing this there is no loss in generality in assuming that $p_n(0)$ is continuous and decreasing. Then $h(u) = \int_0^u p_s(0) \, ds$ satisfies $h(n) \sim g(n)$ as $n \to \infty$. We will simply write $h(u)$ as $g(u)$. (*See*, e.g. Theorem 1.3.3, [BGT].) Actually, this is what is done to prove Theorem 1.3, although we didn't mention it explicitly. Using (2.79) and Proposition 2.3, we see, in the notation of (2.8) and (2.9), that for all n sufficiently large

$$\begin{aligned} \frac{g(n)}{g_1(n)} &= \exp \left(\int_{n/l_2 g(n)}^n \frac{g'(u)}{g(u)} \, du \right) \\ &\leqq \exp \left(\int_{n/l_2 g(n)}^n C \frac{du}{u g(u)} \right) \\ &\leqq \exp \left(\frac{C \log l_2 g(n)}{g_1(n)} \right) \\ &\leqq \exp \left(\frac{C \, l_2 g(n) \log l_2 g(n)}{g(n)} \right). \end{aligned} \tag{2.82}$$

This last term clearly goes to one as n goes to infinity. Thus we see that (1.14) is satisfied by symmetric random walks on Z^2. This enables us to simplify the denominators in the analogues of (1.17) and (1.18) for random walks to get (1.6) and (1.7).

There is another point which we must consider. The hypothesis (1.10), with t replaced by n, of Theorem 1.3 is replaced in Theorem 1.1 by the weaker hypothesis that g is slowly varying at infinity. To verify this we must reconsider the proof of Theorem 1.2. Theorem 1.2 is proved for g satisfying any of the hypothesis (1.14)-(1.16). However, the proofs used when g satisfies (1.14) are relatively simple and it is not difficult to check that (1.10), with t replaced by n, is not used in this case until the final step of the proof, the verification of (2.61). We complete the proof of this Theorem by verifying (2.61) using only the facts that g is slowly varying at infinity and satisfies (1.14).

By (1.14) and (2.70) we have that

$$\begin{aligned} \mathrm{E}^0 \left(a \left(\mathrm{X}_{t_{n-1}}, s_n/2 \log n\right)\right) &\geqq \frac{g\left(t_n/2 \log n\right) - g\left(t_{n-1}\right)}{g\left(t_n/2 \log n\right)} \\ &\geqq (1 - \varepsilon)\left(1 - \frac{1}{\theta}\right) = (1 - \varepsilon') \quad (2.83) \end{aligned}$$

where ε and ε' can be taken to be arbitrarily small for θ sufficiently large. (Note that in this argument we are obtaining lower bounds so θ is taken to be large). Using (2.83) and the obvious fact that $a\left(\mathrm{X}_{t_{n-1}}, s_n/2 \log n\right) \leqq 1$, we get (2.69) for all $n \geqq j$ for all j sufficiently large. This is all we need to use the Paley-Zygmund lemma to obtain (2.61) as in the proof of Theorem 1.2. This completes the proof of Theorem 1.1.

Proof of Theorem 1.4. – This is just Proposition 2.1 of this paper applied to Theorems 1.1 and 1.2 of [MR1] except for one minor point. In [MR1] we required that $\psi(\lambda)$ be regularly varying at zero with index $1 < \beta \leqq 2$. This was not necessary. It would have been enough to simply require that $\kappa(\alpha)$ be regularly varying at zero. We will explain this in the context of Theorem 1.4. We assume that g is regularly varying at infinity with index $0 < 1/\overline{\beta} \leqq 1/2$. This implies, by Theorem 1.72, [BGT] that $g'(t) = p_t(0)$, which is monotonically decreasing, is regularly varying at infinity with index greater than minus one. As we show in (2.49) [MR1]

$$p_t(0) \sim \frac{\Gamma(1 + 1/\beta)}{\pi} \psi^{-1}(1/t) \quad \text{as } t \to \infty. \qquad (2.84)$$

Thus ψ^{-1} and hence ψ is regularly varying at zero. Therefore, it is enough to assume that g is regularly varying at infinity in Theorem 1.4.

Proof of Remark 1.5. – The only point here, other than a change of notation, is explained by (2.11) when g is slowly varying. It is trivial to see that (2.11) also holds when g is regularly varying with index greater than zero.

3. EXAMPLES

We first show that there are many examples of real valued Lévy processes and symmetric random walks on Z^1 which satisfy the hypotheses of Theorems 1.2 and 1.3.

PROPOSITION 3.1. – *Let $h(t)$ be a real valued function which is regularly varying at infinity with index minus one. There exists a Lévy process $\{X(t),\ t \in R^+\}$, with transition probabilities p_t, such that*

$$p_t(0) \sim h(t) \quad as\ t \to \infty \tag{3.1}$$

and similarly for symmetric random walks on Z^1.

Proof. – We will give the proof for Lévy processes. The proof for symmetric random walks is similar. We have

$$\psi(\lambda) = -2\int_0^\infty (1 - \cos \lambda x)\, d\nu[x, \infty) \tag{3.2}$$

where ν a Lévy measure, *i.e.* $-\int_0^\infty (1 \wedge x^2)\, d\nu[x, \infty) < \infty$. Aside from (1.9) we do not impose any conditions on $\psi(\lambda)$ for $\lambda \geqq \lambda_0 > 0$, for some λ_0 small. This is equivalent to saying that, aside from the weak restriction (1.9), there are no restrictions on ν near zero. Indeed for our purposes we need only consider the part of ν on $[x_0, \infty)$, for some $x_0 > 0$. Then by results for characteristic functions, (*see* e.g. [P]), we have

$$\psi(\lambda) \sim C\nu[1/\lambda, \infty). \tag{3.3}$$

Since $\nu[x, \infty)$ can be taken to be any decreasing regularly varying function of index minus one, we can find functions $\psi(\lambda)$ which are asymptotic to any increasing regularly varying function of index one at zero. Furthermore the fact that $\psi(\lambda)$ is asymptotic to an increasing function is not a restriction, because all regularly varying functions of positive index are asymptotic to an increasing function. If ψ is regularly varying at zero so is ψ^{-1}. Thus, since (2.84) also holds for β equal one, (same reference), we see that $p_t(0) \sim \psi^{-1}(1/t)/\pi$ as t goes to infinity. This proves the Proposition since we can always choose ν so that $h_t(0) \sim \psi^{-1}(1/t)/\pi$ as t goes to infinity.

Comparing Propositions 2.14 and 3.1 we see that we get a much richer class of examples for processes on R^1 or Z^1.

The next Proposition will be used to show that we can find Lévy processes for which (1.14) does not hold and (1.15) does.

PROPOSITION 3.2. – *Let*

$$\overline{g}(t) = e^{h(t)\log t} \tag{3.4}$$

where $\lim_{t\to\infty} h(t) = 0$ *and* $\lim_{t\to\infty} h(t) \log t = \infty$. *Assume also that* $h'(u) < 0$ *for* $u \geqq u_0$, *for some* u_0 *sufficiently large and satisfies*

$$|h'(u)| = o\left(\frac{1}{u \log u \log_3 u}\right), \qquad \forall u \geqq u_0. \tag{3.5}$$

Then $\overline{g}$ *is slowly varying at infinity and*

$$\mathrm{H}(t) = \frac{\overline{g}(t/\log_2 \overline{g}(t)}{\overline{g}(t)} \sim e^{-h(t) \log_3 \overline{g}(t)} \quad as\ t \to \infty. \tag{3.6}$$

In particular, if for t sufficiently large,

$$h(t) = (\log_2 t)^{-1} \quad then\ \mathrm{H}(t) \to 1 \quad as\ t \to \infty \tag{3.7}$$

$$h(t) = (\log_3 t)^{-1} \quad then\ \mathrm{H}(t) \to \frac{1}{e} \quad as\ t \to \infty \tag{3.8}$$

and

$$h(t) = (\log_k t)^{-1} \quad then\ \mathrm{H}(t) \to 0 \quad as\ t \to \infty \tag{3.9}$$

for $k > 3$.

Proof. – We will prove (3.6). The proof that $\overline{g}$ is slowly varying at infinity is similar. We have

$$\begin{aligned}\overline{g}(t/\log_2 \overline{g}(t)) &= \exp((\log t - \log_3 \overline{g}(t))\, h(t/\log_2 \overline{g}(t))) \\ &= \exp((\log t - \log_3 \overline{g}(t))\,(h(t) \\ &\quad + h(t/\log_2 \overline{g}(t)) - h(t)).\end{aligned}$$

We see from this that (3.6) holds if

$$\lim_{t\to\infty} |h(t/\log_2 \overline{g}(t)) - h(t)| \log t = 0.$$

This follows from (3.5) since

$$\begin{aligned}|h(t/\log_2 \overline{g}(t)) - h(t)| &\leqq \delta\left(\int_{t/\log_2 \overline{g}(t)}^{t} \frac{du}{u \log u \log_3 u}\right) \\ &\leqq 2\delta\left(\frac{\log_3 \overline{g}(t)}{\log t \log_3 t}\right)\end{aligned}$$

for t sufficiently large, for all $\delta > 0$. The examples in (3.7)-(3.9) follow immediately from (3.6).

Consider $\overline{g}$, as defined in (3.4), for the functions h given in (3.7)-(3.9). It is easy to see that $\overline{g}'$ is regularly varying at infinity with index minus one in all these cases. Thus by Proposition 3.1, in each of these cases, we can find a Lévy process with $p_t(0) \sim \overline{g}'(t)$ and hence with corresponding $g(t) \sim \overline{g}(t)$. We see that (1.15) is satisfied by all the examples whereas (1.14) is only satisfied by the example in (3.7). This argument is also valid for symmetric random walks on Z^1.

We will now consider condition (1.16). Let $f(t) = g(t)/(tg'(t))$. Note that

$$\log\left(\frac{g(t)}{g(1)}\right) = \int_1^t \frac{g'(y)}{g(y)}\,dy = \int_1^t \frac{1}{yf(y)}\,dy.$$

By the representation theorem for slowly varying functions, we see that $\lim_{t\to\infty} f(t) = \infty$. Also

$$\frac{1}{g(t)}\int_1^t \frac{g(y)}{y}\,dy \leqq \frac{1}{g(t)}\int_1^t f(y)\,dg(y) = f(t) - \frac{1}{g(t)}\int_1^t g(y)\,df(y).$$

If $f(t)$ is increasing then (1.16) holds with C = 1. Clearly mild smoothness conditions on $f(t)$ could make the last integral in (3.10) negative but we do not know if (1.16) is true for all slowly varying functions g that are truncated Green's function. We mention (1.16) because it seems to straddle (1.14) and (1.15). By Proposition 3.1, $g'(t)$ can be chosen at will. Hence it is easy to find examples of processes for which (1.16) is satisfied. This argument is also valid for symmetric random walks on Z^1.

Finally, it is easy to see that for symmetric random walks on Z^2, we can take $p_n(0)$ asymptotic to any regularly varying function at infinity of index minus one as long as Proposition 2.14 is satisfied. The simplest way to see this is to take $Y_1 = (\xi_1, \xi_2)$ where ξ_1 and ξ_2 are independent symmetric random variables on Z^1. Then $p_n(0) = p_{n,1}(0)\,p_{n,2}(0)$, where $p_{n,i}(0)$ are the transition probabilities for ξ_i, $i = 1, 2$. Our assertion is obvious now since we can take ξ_1 and ξ_2 to be any random variables in the domain of attraction of the normal.

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