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CONTINUITY AND SINGULARITY OF THE INTERSECTION LOCAL TIME OF STABLE PROCESSES IN $\mathbb{R}^2$

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We show that the planar symmetric stable process X_t of index $\frac{4}{3} < \beta < 2$ has an intersection local time $\alpha(x, \cdot)$ which is weakly continuous in $x \neq 0$, while

$$\alpha\big(x, [0, T]^2\big) \sim \frac{c}{|x|^{2-\beta}}, \quad \text{as } x \to 0.$$

Let X_t be a planar symmetric stable process of index β [see, e.g., Blumenthal and Getoor (1968), pages 19 and 71]. We say that X_t has an intersection local time $\alpha(\cdot, \cdot)$ if there exists a kernel $\alpha(x, B)$ satisfying

$$\int f(x)\alpha(x, B)\, dx = \int_B\int f(X_t - X_s)\, ds\, dt,$$

for all bounded Borel functions f and sets $B \subseteq \mathbb{R}^2_+$.

THEOREM. *If $\frac{4}{3} < \beta < 2$, then X_t has an intersection local time $\alpha(x, \cdot)$ which is weakly continuous in $x \neq 0$, while*

$$\text{(1)} \qquad \alpha\big(x, [0, T]^2\big) - \frac{2c(\beta)T}{|x|^{2-\beta}} \textit{ has an extension continuous for all } x,$$

where

$$c(\beta) = \frac{1}{2^{\beta}\pi}\Gamma\left(\frac{2-\beta}{2}\right)\Big/\Gamma(\beta/2).$$

It is interesting to note that the condition $\beta > \frac{4}{3}$ is precisely the condition for the *existence* of triple points [Taylor (1966)].

Our proof follows the ideas of Le Gall (1985), who proved a similar result for Brownian motion, a result which goes back to Varadhan (1969). See also Rosen (1986b), Yor (1985, 1986) and Dynkin (1985). Some work on intersection local times for Lévy processes is contained in Shieh (1985).

To prove our assertions, define for $\varepsilon > 0$,

$$\alpha_\varepsilon(x, B) = \int_B\int f_\varepsilon(X(s, t) - x)\, ds\, dt,$$

where $X(s, t) = X_t - X_s$ and $f_\varepsilon(x)$ is the transition density function for our process. $\alpha_\varepsilon(x, B)$ is clearly continuous in all parameters as long as $\varepsilon > 0$.

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As in the case of Brownian motion, it is easy to show (see below) that

$$\mathbb{E}_z(\alpha_\varepsilon(x, B) - \alpha_{\varepsilon'}(x', B))^p \le c_0(\gamma)|(\varepsilon, x) - (\varepsilon', x')|^{\gamma p}, \tag{2}$$

for all Borel sets $B \subseteq A_1^1$, where

$$A_k^n = [(2k-2)2^{-n}, (2k-1)2^{-n}] \times [(2k-1)2^{-n}, (2k)2^{-n}],$$

and all $\gamma < 1 - 1/\beta$.

Because of the scaling $X_{\lambda t} =_{\mathscr{L}} \lambda^{1/\beta} X_t$, we have $f_{\lambda\varepsilon}(x) = (1/\lambda^{2/\beta}) f_\varepsilon(x/\lambda^{1/\beta})$ and therefore

$$\alpha_\varepsilon(x, B) = 2^{-n(2-2/\beta)} \alpha_{2^n\varepsilon}(2^{n/\beta}x, 2^n B), \tag{3}$$

so that from (2), for all $B \subseteq A_k^{n+1}$,

$$\mathbb{E}(\alpha_\varepsilon(x, B) - \alpha_{\varepsilon'}(x', B))^p \le c_0(\gamma) 2^{-np(2-2/\beta-\gamma)} |(\varepsilon, x) - (\varepsilon', x')|^{\gamma p}. \tag{4}$$

We now define

$$\gamma_\varepsilon(x, B) = \{\alpha_\varepsilon(x, B)\}, \tag{5}$$

where $\{Y\} = Y - \mathbb{E}(Y)$, and following Le Gall (1985) we write, for

$$B \subseteq \{(s, t) | 0 \le s \le t \le 1\},$$

$$\gamma_\varepsilon(x, B) = \sum_{n, k} \gamma_\varepsilon(x, B \cap A_k^n),$$

and use (4) to establish

$$\mathbb{E}\left(\sum_{k=1}^{2^n} \gamma_\varepsilon(x, B \cap A_k^{n+1}) - \gamma_{\varepsilon'}(x', B \cap A_k^{n+1}) \right)^p \le c_0(\gamma) 2^{np/2} 2^{-np(2-2/\beta-\gamma)} |(\varepsilon, x) - (\varepsilon', x')|^{\gamma p}, \tag{6}$$

so that

$$\|\gamma_\varepsilon(x, B) - \gamma_{\varepsilon'}(x', B)\|_p \le c|(\varepsilon, x) - (\varepsilon', x')|^\gamma, \tag{7}$$

as long as $\frac{1}{2} - 2 + 2/\beta + \gamma < 0$.

It is possible to choose $\gamma > 0$ small satisfying this condition if $\frac{1}{2} - 2 + 2/\beta < 0$, i.e., $\beta > \frac{4}{3}$. (7) was established for $B \subseteq \{(s, t)|0 \le s \le t \le 1\}$, but by symmetry it is clearly true (with a different c) for any bounded Borel set.

Kolmogorov's theorem now shows that locally

$$|\gamma_\varepsilon(x, B) - \gamma_{\varepsilon'}(x', B)| \le c_w |(\varepsilon, x) - (\varepsilon', x')|^\delta, \qquad \varepsilon, \varepsilon' > 0. \tag{8}$$

This assures us of the existence of a continuous limit

$$\gamma(x, B) = \lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(x, B).$$

Let $g(x)$ be any continuous function of compact support away from $x = 0$. The locally uniform convergence of (8) shows

$$
\begin{aligned}
\int g(x)\gamma(x,B)\,dx &= \lim_{\varepsilon\downarrow 0}\int g(x)\{\alpha_\varepsilon(x,B)\}\,dx \\
&= \lim_{\varepsilon\downarrow 0}\int_B\int\{f_\varepsilon * g(X(s,t))\}\,ds\,dt \\
&= \int_B\int\{g(X(s,t))\}\,ds\,dt \\
&= \int_B\int g(X(s,t))\,ds\,dt - \int g(x)\mathscr{U}_B(x)\,dx,
\end{aligned}
\tag{9}
$$

where

$$
\mathscr{U}_B(x) = \int_B\int f_{|t-s|}(x)\,ds\,dt \le c\int_0^\infty f_s(x)\,ds = \frac{\bar{c}}{|x|^{2-\beta}}
\tag{10}
$$

is easily seen to be continuous in $x \neq 0$. Thus

$$
\alpha(x,B) \underset{\text{def}}{\doteq} \gamma(x,B) + \mathscr{U}_B(x)
\tag{11}
$$

is continuous for $x \neq 0$ and satisfies

$$
\int g(x)\alpha(x,B)\,dx = \int g(X(s,t))\,ds\,dt,
$$

for g supported away from $x = 0$, and hence for all continuous compactly supported g [$\mathscr{U}_B(x)$ is integrable at $x = 0$]. (We did not use Le Gall's Theorem 1.1 since it rests on path continuity.)

We note that for $x \neq 0$,

$$
\begin{aligned}
\mathscr{U}_{[0,T]^2}(x) &= \int_0^T\int_0^T f_{|t-s|}(x)\,ds\,dt \\
&= 2\int_0^T\int_0^t f_s(x)\,ds\,dt \\
&= 2T\mathscr{U}(x) - 2\int_0^T\int_t^\infty f_s(x)\,ds\,dt,
\end{aligned}
$$

where

$$
\mathscr{U}(x) = \int_0^\infty f_s(x)\,ds = \frac{c(\beta)}{|x|^{2-\beta}}
$$

is the potential, and

$$
\begin{aligned}
\int_0^T\int_t^\infty f_s(x)\,ds\,dt &\le \int_0^T\int_t^\infty f_s(0)\,ds\,dt \\
&= \int_0^T\left(\int_t^\infty \frac{ds}{s^{2/\beta}}\right)dt = \int_0^T \frac{dt}{t^{2/\beta-1}} < \infty,
\end{aligned}
$$

whenever $1 < \beta < 2$.

Finally, to produce a version of $\alpha(x, B)$ which is a kernel, and which is weakly continuous in $x \neq 0$, we start off in place of (2) with

$$\mathbb{E}_z\big(\alpha_\varepsilon(x, B) - \alpha_{\varepsilon'}(x', B')\big)^p \leq c_1(\gamma)|(\varepsilon, x, B) - (\varepsilon', x', B')|^{\gamma p}, \tag{12}$$

for sufficiently small γ, and all rectangles $B, B' \subseteq A_1^1$, where we identify $B = [a, b] \times [c, d]$ with (a, b, c, d), and proceed as above. See, e.g., Rosen (1986a).

For the convenience of the reader we sketch the proof of (12). As in Rosen (1986a) it will be sufficient to establish a uniform bound for all $B \subseteq A_1^1$,

$$\mathbb{E}\big(\alpha_\varepsilon(x, B)^p\big) \leq c. \tag{13}$$

To this end we use the Fourier representation

$$f_\varepsilon(x) = \frac{1}{(2\pi)^2}\int e^{iux}e^{-\varepsilon|u|^\beta}\, du$$

to find, using independence, that

$$\begin{aligned}\mathbb{E}\big(\alpha_\varepsilon(x, B)^p\big) &= \frac{1}{(2\pi)^{p2}}\int_{B^p}\int e^{\Sigma ixu_j - \varepsilon|u_j|^\beta}\mathbb{E}\big(e^{i\Sigma u_j X(s_j, t_j)}\big)\\ &= \frac{1}{(2\pi)^{p2}}\int_{B^p}\int e^{\Sigma ixu_j - \varepsilon|u_j|^\beta}\mathbb{E}\big(e^{i\Sigma u_j X(s_j, 1/2)}\big)\mathbb{E}\big(e^{i\Sigma u_j X(1/2, t_j)}\big).\end{aligned} \tag{14}$$

We write

$$\sum u_j X\big(s_j, \tfrac{1}{2}\big) = \sum_{j=1}^{p} v_j X(r_j, r_{j+1}),$$

$$\sum u_j X\big(\tfrac{1}{2}, t_j\big) = \sum_{j=0}^{p-1} w_j X(q_j, q_{j+1}),$$

where $r_1, \dots, r_m$ are the s_i's relabeled so that $r_1 \leq r_2 \leq \cdots \leq r_m \leq r_{m+1} \doteq \frac{1}{2}$ and $v_j = \sum_{l:\ s_l \leq r_j} u_l$; in particular, the v_j's span $\mathbb{R}^{2p}$. The q_j's and w_j's are defined analogously. Using (14) and the simple bound

$$\int_0^1 e^{-r|v|^\beta}\, dr \leq \frac{c}{1 + |v|^\beta},$$

we have the uniform bound

$$\begin{aligned}\mathbb{E}\big(\alpha_\varepsilon(x, B)^p\big) &\leq c\int \prod \frac{1}{1 + |v_j|^\beta} \prod \frac{1}{1 + |w_j|^\beta}\, du\\ &\leq c\left\|\prod \frac{1}{1 + |v_j|^\beta}\right\|_2 \left\|\prod \frac{1}{1 + |w_j|^\beta}\right\|_2,\end{aligned} \tag{15}$$

which is finite if $\beta > 1$.

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