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LAPLACE'S METHOD FOR GAUSSIAN INTEGRALS WITH AN APPLICATION TO STATISTICAL MECHANICS

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For a new class of Gaussian function space integrals depending upon $n \in \{1, 2, \cdots\}$, the exponential rate of growth or decay as $n \to \infty$ is determined. The result is applied to the calculation of the specific free energy in a model in statistical mechanics. The physical discussion is self-contained. The paper ends by proving upper bounds on certain probabilities. These bounds will be used in a sequel to this paper, in which asymptotic expansions and limit theorems will be proved for the Gaussian integrals considered here.

I. Introduction. Let $g(y)$ be a continuous function on $\mathbb{R}$ which tends to $+\infty$ sufficiently fast as $|y| \to \infty$. Laplace's method yields [Erdélyi (1956) Section 2.4]

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{\mathbb{R}} \exp[-ng(y)]\, dy = -\inf_{y\in\mathbb{R}} [g(y)].$$

One of the purposes of this paper is to extend Laplace's method to the class of Gaussian integrals on $C[0, 1]$ considered in Theorem 1.1. In the Appendix, we show how to modify our arguments to handle analogous integrals on a Hilbert space.

$C[0, 1]$ denotes the space of real-valued continuous functions $Y(t)$, $t \in [0, 1]$, endowed with the supremum norm. Let P be a mean zero Gaussian measure on $C[0, 1]$ and $\{P_n\}$ a sequence of mean zero Gaussian measures which tend weakly to P ($P_n \Rightarrow P$). Let $\{F_n\}$ be a sequence of suitably bounded and continuous real-valued functionals on $C[0, 1]$ which tend in a suitable sense to a functional F on $C[0, 1]$. Denote by I the entropy functional of P, defined in (4.1). Except in the Appendix, all integrals with respect to P, $\{P_n\}$ are over $C[0, 1]$ unless otherwise noted. $P_n(\sqrt{n}\,\cdot)$ stands for the measure $P_n(\sqrt{n}\mathscr{S}) := P_n(Y: Y/\sqrt{n} \in \mathscr{S})$ for $\mathscr{S}$ a Borel subset of $C[0, 1]$.

THEOREM 1.1. *Assume that P and $\{P_n\}$ are mean zero Gaussian measures on $C[0, 1]$ such that $P_n \Rightarrow P$ (Hypothesis 3.1). Assume that F and $\{F_n\}$ are real-valued functionals on $C[0, 1]$ which are bounded below in the sense of Hypothesis 3.2 and are continuous in the sense of Hypothesis 3.3; assume that the $\{F_n\}$ tend to F in the sense of Hypothesis 3.4. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \int \exp[-nF_n(Y)]\, dP_n(\sqrt{n}\,Y) = -\inf_{Y\in C[0,1]}[F(Y) + I(Y)]. \tag{1.1}$$

Further, $\inf[F + I]$ *over* $C[0, 1]$ *is finite and is attained at some point in* $C[0, 1]$.

The new element in Theorem 1.1 is the treatment of sequences of measures and functionals. Results of Donsker and Varadhan allow us to change some of the hypotheses on the functionals F, $\{F_n\}$ in Theorem 1.1. This alternate form of Theorem 1.1 will be stated below in Theorem 1.4.

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The present paper was inspired by a model in statistical mechanics, called the circle model. We first define the circle model, then show how Theorem 1.1 yields useful information about it. The physical content of the following definitions will be discussed in Section II. (Readers interested only in Theorem 1.1 and its proof can skip Section II with no loss in continuity.)

Let $\mathscr{Y}$ be the family of all continuous, real-valued functions defined on a circle of circumference one. We identify $\mathscr{Y}$ with the closed subspace of $C[0, 1]$ consisting of all functions $Y(t)$, $0 \leq t \leq 1$, which satisfy $Y(0) = Y(1)$. A function $J(s, t)$, $s, t \in [0, 1]$, is said to be a covariance function in $\mathscr{Y}$ if J is the covariance function of a mean zero Gaussian measure on $\mathscr{Y}$. Now let $\{H_n;\, n = 1, 2, \cdots\}$ be a sequence of functions in $\mathscr{Y}$ and $\{J_n;\, n = 1, 2, \cdots\}$ a sequence of covariance functions in $\mathscr{Y}$. Let ρ be a Borel probability measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \exp(cx^2)\, d\rho(x) < \infty \qquad \text{for all} \quad c \geq 0. \tag{1.2}$$

The circle model is defined by the sequence of probability measures $\{\Gamma_n;\, n = 1, 2 \cdots\}$ on $\{\mathbb{R}^n;\, n = 1, 2, \cdots\}$, where for Σ a Borel subset of $\mathbb{R}^n$

$$\Gamma_n(\Sigma) := \frac{\int_{\Sigma} \exp\left[\frac{1}{2} \sum_{i,j=1}^n \frac{1}{n} J_n\left(\frac{i}{n}, \frac{j}{n}\right) X_i X_j + \sum_{i=1}^n H_n\left(\frac{i}{n}\right) X_i\right] \prod_{i=1}^n d\rho(X_i)}{Z_n}. \tag{1.3}$$

Z_n is the normalization constant

$$Z_n := \int_{\mathbb{R}^n} \exp\left[\frac{1}{2} \sum_{i,j=1}^n \frac{1}{n} J_n\left(\frac{i}{n}, \frac{j}{n}\right) X_i X_j + \sum_{i=1}^n H_n\left(\frac{i}{n}\right) X_i\right] \prod_{i=1}^n d\rho(X_i), \tag{1.4}$$

called the *canonical partition function*. The integrals in (1.3)–(1.4) converge because of (1.2). A quantity of considerable physical importance is the *specific* (*Helmholtz*) *free energy*, defined by $f := -\lim_{n\to\infty} n^{-1} \ln Z_n$. Under the hypothesis that the $\{H_n\}$ and $\{J_n\}$ converge suitably as $n \to \infty$, Theorem 1.2 below gives f explicitly.

The connection between Theorem 1.1 and the circle model arises because the canonical partition function Z_n can be represented as a Gaussian integral of the form considered in (1.1). For y real, define

$$\phi(y) := \ln \int_{\mathbb{R}} \exp(yx)\, d\rho(x), \tag{1.5}$$

and let P_n be the mean zero Gaussian measure on $\mathscr{Y}$ with covariance function J_n. Then

$$Z_n = \int_{\mathscr{Y}} \exp(-nF_n(Y))\, dP_n(\sqrt{n}\, Y), \tag{1.6}$$

where

$$F_n(Y) := -\frac{1}{n} \sum_{j=1}^n \phi\left(Y\left(\frac{j}{n}\right) + H\left(\frac{j}{n}\right)\right). \tag{1.7}$$

To see this, note that for $s, t \in [0, 1]$, $J_n(s, t) = E_n\{Y(s)Y(t)\}$, where E_n denotes expectation with respect to P_n. Then

$$\begin{aligned} Z_n &= \int_{\mathbb{R}^n} E_n\left[\exp\left\{\frac{1}{\sqrt{n}} \sum_{i=1}^n Y\left(\frac{i}{n}\right) X_i + H_n\left(\frac{i}{n}\right) X_i\right\}\right] \prod_{i=1}^n d\rho(X_i) \\ &= E_n\left[\exp\left\{\sum_{i=1}^n \phi\left(\frac{1}{\sqrt{n}} Y\left(\frac{i}{n}\right) + H_n\left(\frac{i}{n}\right)\right)\right\}\right], \end{aligned}$$

as claimed.

We next derive the specific free energy for the circle model as an application of Theorem 1.1. The fact that the specific free energy equals the infimum of a functional on $\mathscr{Y}$ is related to the Gibbs variational principle, which is well-known in statistical mechanics (Ruelle (1969) Section 7.4, Simon (1979) Section 19).

THEOREM 1.2. *Assume that in* (1.4) $H_n \in \mathscr{Y}$ *and* J_n *is the covariance function of a mean zero Gaussian measure* P_n *on* $\mathscr{Y}$. *Assume that there exists a function* $H \in \mathscr{Y}$ *and a mean zero Gaussian measure* P *on* $\mathscr{Y}$ *such that* $H_n \to H$ *uniformly and* $P_n \Rightarrow P$. *Define the functional* F *on* $\mathscr{Y}$ *by*

$$F(Y) := -\int_0^1 \phi(Y(u) + H(u))\, du, \tag{1.8}$$

where ϕ *is defined in* (1.5). *Then the specific free energy* f *of the circle model is given by*

$$f := -\lim_{n\to\infty} \frac{1}{n} \ln Z_n = \inf_{Y\in\mathscr{Y}} [F(Y) + I(Y)],$$

where I *is the entropy functional of* P. *Further,* $\inf(F + I)$ *over* $\mathscr{Y}$ *is finite and is attained at some point of* $\mathscr{Y}$.

In future work, we plan to prove asymptotic expansions and limit theorems for the integrals in (1.1). When specialized to the circle model, these results yield detailed information on the asymptotic behavior of the sum of random variables with joint distributions given by (1.3) [Ellis-Rosen (1979)]. This sum is called the total spin and is an important physical quantity. In Section V, we prove the following crucial estimate needed for these asymptotic expansions and limit theorems. It will be proved as a consequence of Theorem 1.1 and Varadhan (1966) Section 3.

THEOREM 1.3. *Let* P, $\{P_n\}$, F, $\{F_n\}$ *satisfy the hypotheses of Theorem* 1.1. *Define probability measures* (*for sufficiently large* n)

$$Q_n(\mathscr{S}) := \frac{\int_{\mathscr{S}} \exp[-nF_n(Y)]\, dP_n(\sqrt{n}\,Y)}{\int \exp[-nF_n(Y)]\, dP_n(\sqrt{n}\,Y)} \tag{1.9}$$

for $\mathscr{S}$ *a Borel subset of* $C[0, 1]$. *Assume that* $\mathscr{A}$ *is a closed subset of* $C[0, 1]$ *such that for some* $\delta > 0$

$$\inf_{Y\in\mathscr{A}}[F(Y) + I(Y)] - \min_{Y\in C[0,1]}[F(Y) + I(Y)] > \delta. \tag{1.10}$$

Then for all sufficiently large n

$$Q_n(\mathscr{A}) \le e^{-n\delta}. \tag{1.11}$$

We discuss results in the literature related to ours. In the special cases where all the P_n's equal P and all the F_n's equal F, (1.1) has been proved by a number of people (Schilder (1966), Pincus (1968), Donsker-Varadhan (1976), Simon (1979) Section 18). All these sources but the third impose strong conditions upon the covariance function of P. Section IV of the present paper proves Theorem 1.1 by extending the methods of Simon (1979) Section 18 but with only the minimal hypothesis that $P_n \Rightarrow P$; otherwise, P, $\{P_n\}$ are arbitrary. Our extension is highly non-trivial. For special cases, we have a more elementary proof of Theorem 1.1 than the proof presented here; see Ellis-Rosen (1979) Section V.

A possible alternate approach to Theorem 1.1 is contained in Varadhan (1966) Section 3. Assume that for any closed subset $\mathscr{A}$ of $C[0, 1]$

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{A}) \le -\inf_{Y\in\mathscr{A}} I(Y) \tag{1.12}$$

and for any open subset $\mathscr{B}$ of $C[0, 1]$

$$\text{(1.13)} \qquad \liminf_{n\to\infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{B}) \geq -\inf_{Y\in\mathscr{B}}(I(Y)).$$

Then it is proved that for suitable F, $\{F_n\}$, (1.1) holds. The bounds (1.12)–(1.13) are derived in Donsker-Varadhan (1976) with $C[0, 1]$ replaced by any separable Banach space $\mathscr{X}$ and $P_1 = P_2 = \cdots = P$, any mean-zero Gaussian measure on $\mathscr{X}$. However, since these bounds have not yet been proved for a sequence of Gaussian measures, (1.1) requires separate proof.

Our proof of (1.1) has the virtues of being direct and almost completely self-contained. Besides, (1.1) can be shown to yield the bounds (1.12)–(1.13). In fact, (1.12), which will be useful in the proof of Theorem 1.3, is derived directly from (1.1) in Section V. We are therefore in the following interesting situation. Although (1.12)–(1.13) have not been proved directly for a sequence $\{P_n\}$ tending weakly to P, they can be derived indirectly via (1.1). By Varadhan (1966) Section 3 and Donsker-Varadhan (1976) Lemma 6.4, we have the following alternate version of Theorem 1.1.

THEOREM 1.4 *Assume that P and $\{P_n\}$ are mean zero Gaussian measures on $C[0, 1]$ such that $P_n \Rightarrow P$. Then* (1.12)–(1.13) *are valid. Also* (1.1) *holds if the functionals F, $\{F_n\}$ satisfy Hypothesis* 3.4 *and*

$$\text{(1.14)} \qquad \lim_{L\to\infty}\limsup_{n\to\infty} \frac{1}{n} \ln \int_{\{Y:-F_n(Y)\geq L\}} \exp[-nF_n(Y)]\, dP_n(\sqrt{n}Y) = -\infty.$$

The second assertion in Theorem 1.1 *holds if F is lower semicontinuous on $C[0, 1]$. Finally, if $\sigma(s, t)$ denotes the covariance function of P, we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln P_n\{Y: \sup_{0\leq t\leq 1} |Y(t)| \geq \sqrt{n}\} = -\frac{1}{2a}, \qquad \text{where} \quad a := \sup_{0\leq t\leq 1}\sigma(t, t).$$

One can prove that (1.14) holds if for all sufficiently small $\nu > 0$ there exists $C_1 = C_1(\nu) \geq 0$ such that each F_n satisfies (3.2). (This is a stronger form of Hypothesis 3.2 than is needed for Theorem 1.1) That the lower semicontinuity of F implies the second assertion in Theorem 1.1 follows from the proof of Lemma 4.5(d). The last assertion in Theorem 1.4 generalizes a result of Marcus-Shepp (1972).

Section II of this paper gives some physical motivation for the circle model. Section III states the hypotheses for Theorem 1.1 and proves Theorem 1.2. Sections IV and V prove Theorems 1.1 and 1.3, respectively. Section VI extends Theorems 1.1, 1.3, and 1.4 to a separable real Hilbert space.

II. Circle Model in Statistical Mechanics. The circle model is a special case of a class of systems called spin systems, which we define in some generality.

Let Λ be a subset of $\mathbb{R}^d$, $d \in \{1, 2, \cdots\}$, consisting of a finite number of points, called *sites*. A *spin system* on Λ is a special family of random variables $\{X_j^\Lambda; j \in \Lambda\}$. Each X_j^Λ represents the spin, or magnetic moment, of the individual atom at the site j in a magnetic crystal with shape described by Λ. When the crystal is kept at a fixed temperature, the joint distribution of the $\{X_j^\Lambda\}$ is given by

$$\text{(2.1)} \quad d\Gamma_\Lambda(X_1, \cdots, X_{|\Lambda|}) := \frac{1}{Z_\Lambda} \exp\left[\beta\left\{\frac{1}{2}\sum_{i,j\in\Lambda} J_\Lambda(i, j)X_iX_j + \sum_{i\in\Lambda} H_\Lambda(i)X_i\right\}\right] \prod_{i\in\Lambda} d\rho(X_i).$$

This measure is also known as the *Gibbs measure* of the system.

We define the symbols in (2.1). $|\Lambda|$ denotes the number of sites in Λ. The number β is proportional to the inverse absolute temperature, so $\beta > 0$. The symbol ρ denotes a Borel probability measure on $\mathbb{R}$ which satisfies (1.2). For $\beta = 0$, the $\{X_j^\Lambda\}$ are independent random variables, each distributed by ρ. J_Λ is a real-valued function on $\mathbb{R}^d \times \mathbb{R}^d$; if J_Λ is non-negative, we say that J_Λ is *ferromagnetic*. The summand $J_\Lambda(i, j)X_iX_j$ denotes the interaction strength between sites i and j with spins X_i, X_j, respectively. $H_\Lambda(x)$ is a smooth function of $x \in \mathbb{R}^d$; $H_\Lambda(i)$ denotes the strength of an external magnetic field applied at site i. The summand $H_\Lambda(i)X_i$ denotes the interaction strength between the external field and the spin at i. Z_Λ is the normalization constant

$$(2.2) \qquad Z_\Lambda := \int_{\mathbb{R}^{|\Lambda|}} \exp\left[\beta\left\{\frac{1}{2}\sum_{i,j\in\Lambda} J_\Lambda(i,j)X_iX_j + \sum_{i\in\Lambda} H_\Lambda(i)X_i\right\}\right] \prod_{i\in\Lambda} d\rho(X_i),$$

which converges because of (1.2). Z_Λ is called the *canonical partition function*.

Important physical quantities are the *total spin* $S_\Lambda := \sum_{i\in\Lambda} X_i^\Lambda$ and its expected value, $E\{S_\Lambda\}$. Macroscopic properties of the spin system are studied by letting Λ become large in a suitable way; in particular, $|\Lambda| \to \infty$. For example, a useful approach to critical phenomena and phase transitions is to study the asymptotic behavior of S_Λ and $E\{S_\Lambda\}$ as $|\Lambda| \to \infty$. See Cassandro-Jona-Lasinio (1978) for a detailed discussion.

We define the quantities

$$(2.3) \qquad \begin{aligned} \Psi_\Lambda &= \Psi_\Lambda(\beta, H_\Lambda) := -\frac{1}{\beta}\ln Z_\Lambda, \\ f &:= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\ln \Psi_\Lambda = -\frac{1}{\beta}\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\ln Z_\Lambda, \end{aligned}$$

known, respectively, as the *(Helmholtz) free energy* and the *specific free energy* of the system. The limit defining f exists under suitable natural hypotheses on $\{\Lambda\}$, $\{J_\Lambda\}$, $\{H_\Lambda\}$, and ρ, Ruelle (1969). Among other reasons, these quantities are useful because of their relation to S_Λ and $E\{S_\Lambda\}$. For example.

$$\ln E\{\exp(rS_\Lambda)\} = -\beta[\Psi_\Lambda(\beta, H_\Lambda + r) - \Psi_\Lambda(\beta, H_\Lambda)],\ r \text{ real},$$

and if $H_\Lambda = h$, a constant, then

$$E\{S_\Lambda\} = -\beta\frac{\partial}{\partial h}\Psi_\Lambda(\beta, h).$$

Similarly, f is useful in studying the asymptotic behavior of S_Λ and $E\{S_\Lambda\}$ as $|\Lambda| \to \infty$.

The circle model (1.3)–(1.4) fits into the framework of (2.1)–(2.2). Indeed, the sequence of measures (1.3)–(1.4) defines spin systems on the subsets $\Lambda_n := \{j/n; j = 1, \cdots, n\}$, $n = 1, 2, \cdots$, of $[0, 1]$ (alias a circle of circumference one). For notational convenience, we have set $\beta = 1$ in (1.3). We do not require J to be ferromagnetic.

Because of the factor $1/n$ multiplying $J_n(i/n, j/n)$, (1.3)–(1.4) define a so-called mean field type model. The choice $J = 1$ and $H = h$, a constant, defines the Curie-Weiss model, Brout (1968), Kac (1968) Section 3. The asymptotic behavior of S_{Λ_n} as $n \to \infty$ has been studied in great detail for this model (Ellis-Newman (1978a), (1978b), Ellis-Newman-Rosen (1980)).

III. Hypotheses for Theorem 1.1. After some preliminaries, we state our hypotheses for Theorem 1.1, and then we prove Theorem 1.2.

We write L^2 for $L^2[0, 1]$. L^2 and $C[0, 1]$ are regarded as spaces of real-valued functions. We denote by $\|-\|$, $\|-\|_2$, and $(-, -)$ the supremum norm on $C[0, 1]$, the L^2-norm, and the L^2-inner product, respectively. All functions belong to $C[0, 1]$ unless otherwise noted. Given a sequence $\{Y_n\}$ and an element Y, the notation $Y_n \to Y$ means $\|Y - Y_n\| \to 0$. All constants are independent of n unless otherwise noted.

Let P, $\{P_n\}$ be mean zero Gaussian measures on $C[0, 1]$ with covariance functions $\sigma(s, t)$, $\{\sigma_n(s, t)\}$, $0 \le s, t \le 1$. We denote by $\mathscr{J}$, $\{\mathscr{J}_n\}$ the covariance operators on L^2 corresponding to P, $\{P_n\}$. We have for $Y \in L^2$

$$(3.1) \qquad \mathscr{J}Y(s) = \int_0^1 \sigma(s, t)Y(t)\,dt, \qquad \mathscr{J}_nY(s) = \int_0^1 \sigma_n(s, t)Y(t)\,dt.$$

$\mathcal{J}$ and each $\mathcal{J}_n$ are positive, symmetric, trace class operators, Gihman-Skorohod (1974) Theorem 1, page 350. None of these operators need be strictly positive. We denote by $\mathcal{T}$ and $\mathcal{T}_n$ the closed subspaces of L^2 on which $\mathcal{J}$ and $\mathcal{J}_n$, respectively, are strictly positive, and by τ and τ_n the orthogonal projections onto $\mathcal{T}$ and $\mathcal{T}_n$, respectively. The subspaces $\mathcal{T}$ and $\mathcal{T}_n$ are, respectively, invariant subspaces for $\mathcal{J}$ and $\mathcal{J}_n$.

HYPOTHESIS 3.1. *P, $\{P_n\}$ are mean zero Gaussian measures on $C[0, 1]$ such that $P_n \Rightarrow P$.*

HYPOTHESIS 3.2 *F, $\{F_n\}$ are real-valued functionals on $C[0, 1]$. There exist numbers ν and C_1 satisfying $0 < \nu < 1/(2\|\mathcal{J}\|_2)$ and $C_1 \geq 0$ such that for all $Y \in C[0, 1]$ and all sufficiently large n*

$$F(Y) \geq -\nu \|Y\|_2^2 - C_1, \qquad F_n(Y) \geq -\nu \|Y\|_2^2 - C_1. \tag{3.2}$$

HYPOTHESIS 3.3 *There exists $\gamma \in (0, 1]$ and, for each $R > 0$, there exists $C(R) > 0$ such that if $\|X\| \leq R$, $\|Y\| \leq R$, then for all sufficiently large n*

$$|F(X) - F(Y)| \leq C(R)\|X - Y\|_2^\gamma, \qquad |F_n(X) - F_n(Y)| \leq C(R)\|X - Y\|_2^\gamma. \tag{3.3}$$

HYPOTHESIS 3.4 *Let $\mathcal{D}$ be the domain of $\mathcal{J}^{-1/2}$, the inverse of the square root of $\mathcal{J}$ ($\mathcal{D} \subseteq C[0, 1]$ by Lemma 4.3(d)). Given $Y \in \mathcal{D}$ and $\{Y_n\}$ a sequence such that $Y_n \to Y$, then $F_n(Y_n) \to F(Y)$.*

REMARK 3.5. The inequality (3.2) implies that for all n sufficiently large $\int \exp(-nF_n(Y))\, dP_n(\sqrt{n}Y) < \infty$.

PROOF OF THEOREM 1.2. We shall prove Theorem 1.2 as an application of Theorem 1.4. As noted in the Introduction, Theorem 1.4 is a consequence of Theorem 1.1. By Theorem 1.4, it suffices to prove that F and the $\{F_n\}$ satisfy Hypothesis 3.4, that the $\{F_n\}$ satisfy (1.14), and that F is continuous. The functionals F and $\{F_n\}$ are defined in (1.8) and in (1.7), respectively. Concerning Hypothesis 3.4, we have that $|\phi'(y)|$ is bounded for y in a fixed compact set. Hence there exists a constant C (depending on Y, $\{Y_n\}$) so that

$$\begin{aligned} |F(Y) - F_n(Y_n)| &= \left| \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left[\phi\Big(Y(u) + H(u) \Big) - \phi\left(Y_n\left(\frac{j}{n}\right) + H\left(\frac{j}{n}\right)\right)\right] du \right| \\ &\leq C \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left| Y(u) + H(u) - Y\left(\frac{j}{n}\right) - H\left(\frac{j}{n}\right) \right| du. \end{aligned}$$

Since $Y_n \to Y$ and $H_n \to H$ uniformly, we conclude that $|F(Y) - F_n(Y_n)| \to 0$. To prove that the $\{F_n\}$ satisfy (1.14), we first show that for any $\nu > 0$ there exists $C_1 = C_1(\nu) \geq 0$ such that for all $n \in \{1, 2, \cdots\}$ and $Y \in \mathcal{Y}$

$$F_n(Y) \geq -\nu \|Y\|^2 - C_1. \tag{3.4}$$

For any $\nu > 0$, and x, y real, we have $yx \leq \nu y^2/2 + x^2/(2\nu)$. Thus,

$$\phi(y) \leq \frac{\nu}{2} y^2 + B,$$

where $B := \ln \int_{\mathbb{R}} \exp[x^2/(2\nu)]\, d\rho(x)$. B is finite because of (1.2). Thus, for $n \in \{1, 2, \cdots\}$ and $Y \in \mathcal{Y}$

$$\begin{aligned} F_n(Y) &\geq -\frac{\nu}{n} \sum_{j=1}^n \left[Y\left(\frac{j}{n}\right)\right]^2 - \frac{\nu}{n} \sum_{j=1}^n \left[H\left(\frac{j}{n}\right)\right]^2 - B \\ &\geq -\nu \|Y\|^2 - C_1, \end{aligned}$$

where $C_1 := \nu\|H\|^2 + B$. This proves (3.4). To show (1.14), we note that because of (3.4) and (4.4) later, we have for all sufficiently large L

$$(3.5) \qquad P_n\{\sqrt{n}\{Y: -F_n(Y) \geq L\}\} \leq P_n\left\{Y: \|Y\|^2 \geq \frac{nL}{2\nu}\right\} \leq C\exp\left[-n\beta\left(\frac{L}{2\nu}\right)\right],$$

where $C > 0$, $B > 0$. Again by (3.4), the integral in (1.14) is bounded above by

$$e^{nC_1}[P_n\{\sqrt{n}\{Y: -F_n(Y) \geq L\}\}]^{1/2}\left[\int_{\mathscr{Y}} \exp(2\nu\|Y\|^2)\,dP_n(Y)\right]^{1/2}.$$

Now (1.14) follows from (3.5) and (4.5) (ii) below, provided ν is sufficiently small. Since F is obviously continuous on $\mathscr{Y}$, the proof of Theorem 1.2 is complete. □

IV. Proof of Theorem 1.1. We first define the entropy functionals of the measures P, $\{P_n\}$. Let $\mathscr{J}$, $\{\mathscr{J}_n\}$ be the covariance operators corresponding to these measures; see (3.1). We denote by $\sqrt{\mathscr{J}}$, $\{\sqrt{\mathscr{J}_n}\}$ the unique symmetric, positive square roots of $\mathscr{J}$, $\{\mathscr{J}_n\}$. $\sqrt{\mathscr{J}}$ is invertible on a dense subset of $\mathscr{T}$ with inverse $\mathscr{J}^{-1/2}$; each $\sqrt{\mathscr{J}_n}$ is invertible on a dense subset of $\mathscr{T}_n$ with inverse $\mathscr{J}_n^{-1/2}$ ($\mathscr{T}$ and $\mathscr{T}_n$ were defined after (3.1)). We denote the domains of these inverses by $\mathscr{D}$, $\{\mathscr{D}_n\}$, respectively. $\mathscr{D}$ and $\mathscr{D}_n$ are subsets of $C[0, 1]$ (Lemma 4.3(d)). We define the functionals (entropy functionals of P, $\{P_n\}$)

$$(4.1) \qquad I(Y) := \begin{cases} \frac{1}{2}\|\mathscr{J}^{-1/2}Y\|_2^2, & Y \in \mathscr{D}, \\ +\infty, & Y \in C[0, 1]\backslash\mathscr{D}, \end{cases}$$

$$(4.2) \qquad I_n(Y) := \begin{cases} \frac{1}{2}\|\mathscr{J}_n^{-1/2}Y\|_2^2, & Y \in \mathscr{D}_n, \\ +\infty, & Y \in C[0, 1]\backslash\mathscr{D}_n. \end{cases}$$

We also define

$$(4.3) \qquad G(Y) := F(Y) + I(Y), \qquad G_n(Y) := F_n(Y) + I_n(Y).$$

REMARK 4.1. The entropy functional of a Gaussian measure on a general Banach space is defined in [Donsker-Varadhan (1976) Theorem 6.2]. One may check that for $C[0, 1]$, their definition reduces to (4.1). See also Freidlin (1972) and Wentzell (1972).

We denote by $\{\lambda_j^2;\ j = 1, 2, \cdots\}$ and $\{\lambda_{j,n}^2;\ j = 1, 2, \cdots\}$ the eigenvalues of $\mathscr{J}$ and each $\mathscr{J}_n$ and by $\{\zeta_j; j = 1, 2, \cdots\}$ and $\{\zeta_{j,n}; j = 1, 2, \cdots\}$ corresponding orthonormal eigenfunctions. The eigenvalues are repeated according to multiplicity and arranged in decreasing order. We denote by $\{\tilde{\lambda}_j^2;\ j = 1, 2, \cdots, N\}$, $N \leq \infty$, the distinct positive eigenvalues of $\mathscr{J}$; by $\{\tilde{\lambda}_{j,n}^2;\ j = 1, 2, \cdots, N(n)\}$, $N(n) \leq \infty$, the distinct positive eigenvalues of $\mathscr{J}_n$; by Φ_j the orthogonal projection onto the eigenspace corresponding to $\tilde{\lambda}_j^2$; by $\Phi_{j,n}$ the orthogonal projection onto the eigenspace corresponding to $\tilde{\lambda}_{j,n}^2$.

Next, we prove four lemmas which refer in turn to P, $\{P_n\}$; $\mathscr{J}$, $\{\mathscr{J}_n\}$; I, $\{I_n\}$; G, $\{G_n\}$. There are followed by a lower bound (4.19) and an upper bound (4.20) which together give the theorem. Our proof extends that of Simon (1979) Section 18, where (1.1) is proved in the case $F_1 = F_2 = \cdots = F$, $P_1 = P_2 = \cdots = P$.

LEMMA 4.2.

(a) *There exist positive constants β and C so that for all n*

$$(4.4) \qquad P_n\{Y: \|Y\| > a\} \leq Ce^{-\beta a^2}, \qquad \text{all } a > 0.$$

(b) *For each $\mu > 0$ and each sufficiently small $\bar{\mu} > 0$*

$$(4.5) \qquad \text{(i)} \quad \sup_n \int \exp(\mu\|Y\|)\,dP_n(Y) < \infty, \qquad \text{(ii)} \quad \sup_n \int \exp(\bar{\mu}\|Y\|^2)\,dP_n(Y) < \infty.$$

(c) *There exists a constant K so that*

$$(4.6) \qquad |\sigma_n(s, t)| \leq K, \qquad \text{all } n, \quad 0 \leq s, t \leq 1.$$

(d) *We have*

$$\lim_{n\to\infty}\|\mathscr{I}-\mathscr{I}_n\|_2^2 \le \lim_{n\to\infty}\int_0^1\int_0^1 |\sigma(s,t)-\sigma_n(s,t)|^2\,dsdt = 0. \tag{4.7}$$

(e) *Define the incremental variance $\psi_n(s,t)$ for P_n by*

$$\psi_n(s,t) := \int (Y(s)-Y(t))^2\,dP_n(Y) = \sigma_n(s,s)+\sigma_n(t,t)-2\sigma_n(s,t). \tag{4.8}$$

Given $\varepsilon > 0$, there exists $\delta > 0$ so that if $|s-t| < \delta$, then $\psi_n(s,t) < \varepsilon$ for each n.

(f) *The measure $dP_n([I+\sqrt{n}\mathscr{I}_n]\cdot)$ is absolutely continuous with respect to P_n with Radon-Nikodym derivative*

$$\frac{dP_n([I+\sqrt{n}\mathscr{I}_n]\cdot)}{dP_n}(Y) = \det(I+\sqrt{n}\mathscr{I}_n)\exp\left[-\frac{n}{2}(\mathscr{I}_nY,Y)-\sqrt{n}\|Y\|_2^2\right]. \tag{4.9}$$

(g) *Given $\bar{Y}\in\mathscr{D}$ and positive constants θ, C_2, define*

$$\mathscr{K}_n(\theta,C_2) := \{Y:\|\mathscr{I}_n^{1/2}Y-\mathscr{I}^{-1/2}\bar{Y}\|_2<\theta,\quad \|Y\|<C_2\}. \tag{4.10}$$

Then for any $\theta\in(0,1)$, there exists $C_2 = C_2(\theta)$ so that

$$\liminf_{n\to\infty} P_n\{\mathscr{K}_n(\theta,C_2(\theta))\} > 0. \tag{4.11}$$

PROOF.

(a) Let Q be any mean zero Gaussian measure on $C[0,1]$. The proof of Theorem 1.9 in Marcus-Shepp (1972) shows that if $q > ½$ and $S > 0$ have the property that $Q\{Y:\|Y\|\le S\}\ge q$, then for all $a > S$

$$Q\{Y:\|Y\|>a\}\le e^{-\beta a^2}, \tag{4.12}$$

where $\beta = \beta(q) := (24S^2)^{-1}\ln[q/(1-q)]$. We apply this to the present set-up. Fixing $q > ½$, pick $S > 0$ so that

$$P\{Y:\|Y\|\le S\}\ge q,\qquad P\{Y:\|Y\|=S\}=0.$$

Since $P_n\Rightarrow P$, we have

$$P_n\{Y:\|Y\|\le S\}\to P\{Y:\|Y\|\le S\}$$

by Billingsley (1968) Theorem 2.1(v). By increasing S, if necessary, we guarantee that for some $\delta > 0$ sufficiently small and all n

$$P_n\{Y:\|Y\|\le S\}\ge q-\delta > ½. \tag{4.13}$$

Hence (4.12) holds with Q replaced by any P_n with fixed $\beta := \beta(q-\delta)$. The inequality (4.4) follows with this β and $C := \exp(\beta S^2)$.

(b) This is an immediate consequence of (4.4).

(c) Since $|\sigma_n(s,t)|\le\int\|Y\|^2\,dP_n(Y)\le\int\exp(\sqrt{2}\,\|Y\|)\,dP_n(Y)$, this follows from (4.5)(i).

(d) $P_n\Rightarrow P$ implies $\sigma_n\to\sigma$ pointwise. This, (4.6), and the Lebesgue dominated convergence theorem yield (4.7).

(e) By Prohorov's theorem [Billingsley (1968) Theorem 6.2], $P_n\Rightarrow P$ implies that given $\varepsilon\in(0,1)$ there exists a compact set $\mathscr{K}_\varepsilon$ so that $P_n(\mathscr{K}_\varepsilon)\ge 1-\varepsilon$ for all n. By the Arzelà-Ascoli theorem [Billingsley (1968) page 221], the functions in $\mathscr{K}_\varepsilon$ are equicontinuous. Now

$$\psi_n(s,t) = \left\{\int_{\mathscr{K}_\varepsilon}+\int_{C[0,1]\setminus\mathscr{K}_\varepsilon}\right\}(Y(s)-Y(t))^2\,dP_n(Y). \tag{4.14}$$

For the first integral we use the equicontinuity of $Y\in\mathscr{K}_\varepsilon$. We bound the second integral by

$$\sqrt{\varepsilon}\left\{\int (Y(s)-Y(t))^4\,dP_n(Y)\right\}^{1/2}\le\sqrt{\varepsilon}\left\{\int\exp[5\|Y\|]\,dP_n(Y)\right\}^{1/2}. \tag{4.15}$$

The proof is done because of the arbitrariness of ε and (4.5)(i).

(f) The proof of Simon (1979) Lemma 18.6 works here.

(g) We define the subset of $C[0, 1]$

$$\mathscr{K}(\theta, C_2) = \{Y: \|\mathscr{J}^{1/2}Y - \mathscr{J}^{-1/2}\bar{Y}\|_2 < \theta, \quad \|Y\| < C_2\}.$$

In Simon (1979) Lemma 18.9 it is proved that for any $\theta \in (0, 1)$ there exists $C_2 = C_2(\theta)$ so that $P\{\mathscr{K}(\theta/2, C_2)\} > 0$. We claim that for all n sufficiently large $\mathscr{K}_n(\theta, C_2) \supset \mathscr{K}(\theta/2, C_2)$. Given $Y \in \mathscr{K}(\theta/2, C_2)$

$$\begin{aligned}\|\mathscr{J}_n^{1/2}Y - \mathscr{J}^{-1/2}\bar{Y}\|_2 &\le \|\mathscr{J}_n^{1/2} - \mathscr{J}^{1/2}\|_2\|Y\|_2 + \|\mathscr{J}^{1/2}Y - \mathscr{J}^{-1/2}\bar{Y}\|_2 \\ &\le C_2\|\mathscr{J}_n^{1/2} - \mathscr{J}^{1/2}\|_2 + \theta/2.\end{aligned}$$

Since $\|\mathscr{J}_n^{1/2} - \mathscr{J}^{1/2}\|_2 \to 0$ (to be proved in Lemma 4.3(b)), the claim is proved. The set $\mathscr{K}(\theta/2, C_2)$ is open and $P_n \Rightarrow P$, so that by Billingsley (1968) Theorem 2.1(iv)

$$\liminf_{n\to\infty} P_n(\mathscr{K}_n(\theta, C_2)) \ge \liminf_{n\to\infty} P_n(\mathscr{K}(\theta/2, C_2)) \ge P(\mathscr{K}(\theta/2, C_2)) > 0. \qquad \square$$

Part (a) of the next lemma refers to the projections τ, τ_n, which were defined in the third paragraph of Section III.

LEMMA 4.3.

(a) *For each* j, $\lambda_{j,n}^2 \to \lambda_j^2$, $\|\Phi_j - \Phi_{j,n}\|_2 \to 0$, $\|\tau - \tau_n\|_2 \to 0$ *as* $n \to \infty$.

(b) $\|\sqrt{\mathscr{J}} - \sqrt{\mathscr{J}_n}\|_2 \to 0$.

(c) $\dfrac{1}{n} \ln \det(I + \sqrt{n}\mathscr{J}_n) \to 0$.

(d) $\sqrt{\mathscr{J}}$, $\{\sqrt{\mathscr{J}_n}\}$ *map* L^2 *into* $C[0, 1]$, *so that* $\mathscr{D}$, $\{\mathscr{D}_n\}$, *the domains of the inverses of these operators, are subsets of* $C[0, 1]$.

(e) *For all* $Y \in L^2$

$$\text{(4.16)} \quad \text{(i) } \|\mathscr{J}_nY\| \le K\|Y\|_2, \quad \text{(ii) } \|\sqrt{\mathscr{J}_n}Y\| \le \sqrt{K}\|Y\|_2, \quad \text{(iii) } \|\mathscr{J}_nY\| \le \sqrt{K}\|\sqrt{\mathscr{J}_n}Y\|_2,$$

where K *is the constant in* (4.7). *The same bounds hold with* $\mathscr{J}_n$ *replace by* $\mathscr{J}$.

PROOF.

(a)–(b) $I - \tau_n$ is the spectral projection onto the nullspace of $\mathscr{J}_n$; similarly for $I - \tau$. Since $\|\mathscr{J} - \mathscr{J}_n\|_2^2 \to 0$ by (4.7), (a) follows from Riesz-Sz.-Nagy (1955) pages 370–373 and (b) from (a).

(c) By Gohberg-Kreĭn (1969) Theorem III. 8.5 and page 114 and by (4.7), we have

$$\operatorname{Tr}\mathscr{J}_n = \sum_{j=1}^{\infty} \lambda_{j,n}^2 = \int_0^1 \sigma_n(s, s)\, ds \le K,$$

where Tr denotes trace. Thus, since $\mathscr{J}_n$ is positive,

$$0 \le \frac{1}{n} \ln \det(I + \sqrt{n}\mathscr{J}_n) = \frac{1}{n} \ln \prod_{j=1}^{\infty} (1 + \sqrt{n}\lambda_{j,n}^2) \le \frac{1}{n} \ln e^{\Sigma\sqrt{n}\lambda_{j,n}^2} = \frac{\operatorname{Tr}\mathscr{J}_n}{\sqrt{n}} \to 0.$$

(d)–(e) These are proved in Simon (1979) Lemma 18.4. $\square$

We next prove some facts about the functionals I, $\{I_n\}$.

LEMMA 4.4.

(a) *Let* Y, $\{Y_n\}$ *be functions in* $C[0, 1]$. *If* $Y_n \to Y$, *then* $\liminf_{n\to\infty} I_n(Y_n) \ge I(Y)$. *In particular,* I *is lower semicontinuous on* $C[0, 1]$.

(b) *Let* $\{Y_n\}$ *be a sequence in* $C[0, 1]$. *If* $\sup I_n(Y_n) < \infty$, *then* $\{Y_n\}$ *has a convergent subsequence. In particular, for any* $L \in (0, \infty)$, *the set* $\{Y: I(Y) \le L\}$ *is compact.*

(c) *If* $I(Y) < \infty$, *then there exists a sequence* $\{Y^{(n)}\}$ *with the properties that* $Y^{(n)} \in \mathscr{D}_n$, $Y^{(n)} \to Y$, $I_n(Y^{(n)}) \le I(Y)$, $I_n(Y^{(n)}) \to I(Y)$.

PROOF.

(a) Since $\| Y - Y_n\|_2 \le \| Y - Y_n\| \to 0$, Lemma 4.3(a) implies $\|\Phi_{j,n} Y_n\|_2 \to \|\Phi_j Y\|_2$. Since $\tilde{\lambda}_{j,n}^2 \to \tilde{\lambda}_j^2$, Fatou's lemma implies

$$\liminf I_n(Y_n) = \liminf \tfrac{1}{2} \textstyle\sum_j \tilde{\lambda}_{j,n}^{-2} \|\Phi_{j,n} Y_n\|_2^2$$
$$\ge \tfrac{1}{2} \textstyle\sum_j \tilde{\lambda}_j^{-2} \|\Phi_j Y\|_2^2 = I(Y).$$

The second assertion follows from the first with all the I_n's replaced by I.

(b) Let $D := 2 \sup I_n(Y_n) < \infty$ and define $X_n := \mathscr{I}_n^{-1/2} Y_n$. By Mercer's theorem [Riesz-Nagy (1955) Section 98], we have for each $s, t \in [0,$

$$\sigma_n(s, t) = \textstyle\sum_j \lambda_{j,n}^2 \zeta_{j,n}(s)\zeta_{j,n}(t).$$

If $X_n(t) = \sum_j \beta_{j,n}\zeta_{j,n}(t)$, $\beta_{j,n} := (X_n, \zeta_{j,n})$, then

$$| Y_n(t)|^2 = |(\sqrt{\mathscr{I}_n} X_n)(t)|^2 \le |\textstyle\sum_j \beta_{j,n}\lambda_{j,n}\zeta_{j,n}(t)|^2$$
$$\le \|X_n\|_2^2 \textstyle\sum_j \lambda_{j,n}^2[\zeta_{j,n}(t)]^2 \le D\sigma_n(t, t).$$

Similarly,

$$| Y_n(s) - Y_n(t)|^2 \le \|X_n\|_2^2\{\sigma_n(s, s) + \sigma_n(t, t) - 2\sigma_n(s, t)\} \le D\psi_n(s, t).$$

The first part now follows from Lemma 4.2(d), (e), and the Arzelà-Ascoli theorem. The second assertion follows from the fact that the set $\{Y: I(Y) \le L\}$ is conditionally compact and closed. The conditional compactness follows from the first part with all the I_n's replaced by I; the closure follows from the lower semicontinuity of I.

(c) Define the sequence $Y^{(n)} := \sqrt{\mathscr{I}_n}\mathscr{I}^{-1/2} Y$. We first verify the statements about $I_n(Y^n)$, then prove $Y^{(n)} \to Y$. We have

$$I_n(Y^{(n)}) = \tfrac{1}{2} \|\tau_n \mathscr{I}^{-1/2} Y\|_2^2 \le \tfrac{1}{2} \|\mathscr{I}^{-1/2} Y\|_2^2 = I(Y)$$

since τ_n is an orthogonal projection. Also, $I_n(Y^{(n)}) \to I(Y)$ since $\|\tau - \tau_n\|_2 \to 0$ (Lemma 4.3(a)) and $\tau \mathscr{I}^{-1/2} Y = \mathscr{I}^{-1/2} Y$. Since $Y \in \mathscr{D}$, $Y = \sqrt{\mathscr{I}}\mathscr{I}^{-1/2} Y$. Thus,

$$\| Y - Y^{(n)}\|_2 = \|(\sqrt{\mathscr{I}} - \sqrt{\mathscr{I}_n})(\mathscr{I}^{-1/2} Y)\|_2 \le \|\sqrt{\mathscr{I}} - \sqrt{\mathscr{I}_n}\|_2 \|\mathscr{I}^{-1/2} Y\|_2,$$

which tends to zero as $n \to \infty$ by Lemma 4.3(b). We prove that any subsequence $\{Y^{(n')}\}$ has a subsubsequence which tends to Y in $C([0, 1]$. Then $Y^{(n)} \to Y$ and $\{Y^{(n)}\}$ is the desired sequence. The statement about any subsequence $\{Y^{(n')}\}$ follows from $\| Y - Y^{(n')}\|_2 \to 0$, $I_{n'}(Y^{n'}) \le I(Y) < \infty$ for all n', and part (b) of this lemma. □

We now prove facts about the functionals G, $\{G_n\}$ defined in (4.3).

LEMMA 4.5.

(a) *Let Y be an element and $\{Y_n\}$ a sequence of elements in $C[0, 1]$. If $Y_n \to Y$ and $I(Y) < \infty$, then* $\liminf_{n\to\infty} G_n(Y_n) \ge G(Y)$. *In particular, G is lower semicontinuous on $\mathscr{D}$.*

(b) *Given $I(Y) < \infty$, let $\{Y^{(n)}\}$ be the sequence constructed in Lemma* 4.4(c). *Then* $G_n(Y^{(n)}) \to G(Y)$.

(c) *G is finite on $\mathscr{D}$ and G_n is finite on $\mathscr{D}_n$. There exist constants $\delta > 0$, $C_1 > 0$, so that*

$$G(Y) \ge \delta I(Y) - C_1, \qquad Y \in \mathscr{D}, \tag{4.17}$$

and for all sufficiently large n

$$G_n(Y) \ge \delta I_n(Y) - C_1, \qquad Y \in \mathscr{D}_n. \tag{4.18}$$

(d) Inf *G over $C[0, 1]$ is finite;* inf *G_n over $C[0, 1]$ is finite for all sufficiently large n. There exists at least one point $Y^* \in \mathscr{D}$ and for all sufficiently large n, at least one point $Y_n^* \in \mathscr{D}_n$ so that*

$$G(Y^*) = \inf_{Y\in C[0,1]} G(Y), \qquad G_n(Y_n^*) = \inf_{Y\in C[0,1]} G_n(Y).$$

(e) $G_n(Y_n^*) \to G(Y^*)$.

PROOF.

(a)–(b) These follow from Hypothesis 3.4 and Lemma 4.4(a), (c).

(c) To prove (4.18), we have from (3.2) for $Y \in \mathscr{D}_n$, n sufficiently large,

$$G_n(Y) \geq -\nu \| Y \|_2^2 - C_1 + I_n(Y) \geq I_n(Y)[1 - 2\nu \|\mathscr{J}_n\|_2] - C_1.$$

Since by (4.7) $2\nu \|\mathscr{J}_n\|_2 \to 2\nu \|\mathscr{J}\|_2 < 1$, (4.18) follows. The bound (4.17) is proved similarly.

(d) We prove the statement for G; G_n is handled similarly. Let $U := \inf G$ over $C[0, 1]$; $U \geq -C_1 > -\infty$ by (4.17). Define $\mathscr{F} := \{Y: Y \in \mathscr{D}, G(Y) \leq U + 1\}$. Since G is lower semicontinuous on $\mathscr{D}$, $\mathscr{F}$ is closed. For $Y \subset \mathscr{F}$, $I(Y) \leq \delta^{-1}(U + 1 + C_1)$ by (4.17), so $\mathscr{F}$ is compact by Lemma 4.4(b). A lower semicontinuous function achieves its absolute minimum on a compact set [Berger (1977) Theorem 6.1.1], so we are done since $\inf_{Y \in C[0,1]}[G(Y)] = \inf_{Y \in \mathscr{F}}[G(Y)]$.

(e) We first prove $\limsup_{n\to\infty} G_n(Y_n^*) \leq G(Y^*)$. This follows from the bound $G_n(Y_n^*) \leq G_n((Y^*)^{(n)})$ and part (b) of this lemma, where $\{(Y^*)^{(n)}\}$ is the sequence constructed in Lemma 4.4(c) for $Y := Y^*$. In order to prove $\liminf_{n\to\infty} G_n(Y_n^*) \geq G(Y^*)$, it suffices to prove that any infinite subsequence $\{Y_{n'}^*\}$ has a subsubsequence $\{Y_{n''}^*\}$ so that $\liminf G_{n''}(Y_{n''}^*) \geq G(Y^*)$. By Hypothesis 3.4, $F_{n'}(0) \to F(0)$. Thus,

$$\sup G_{n'}(Y_{n'}^*) \leq \sup G_{n'}(0) = \sup F_{n'}(0) < \infty.$$

By (4.18), $\sup I_{n'}(Y_{n'}) < \infty$. By Lemma 4.4(b), there exists a subsubsequence $\{Y_{n''}^*\}$ and an element $\bar{Y} \in C[0, 1]$ so that $Y_{n''}^* \to \bar{Y}$. By Lemma 4.4(a), $I(\bar{Y}) < \infty$. Hence, by part (a) of the present lemma, $\liminf G_{n''}(Y_{n''}^*) \geq G(\bar{Y}) \geq G(Y^*)$, as required.

We now prove Theorem 1.1. The second assertion of the theorem is proved in Lemma 4.5(d). We prove (1.1). Define the functional

$$\begin{aligned}\Delta_n(Y) &:= F_n\left(\frac{Y}{\sqrt{n}} + \mathscr{J}_n Y\right) + \frac{1}{\sqrt{n}} \| Y \|_2^2 + \frac{1}{2} (\mathscr{J}_n Y, Y) \\ &= G_n(\mathscr{J}_n Y) + F_n\left(\frac{Y}{\sqrt{n}} + \mathscr{J}_n Y\right) - F_n(\mathscr{J}_n Y) + \frac{1}{\sqrt{n}} \| Y \|_2^2.\end{aligned}$$

Changing variables $Y \to (I + \sqrt{n}\, \mathscr{J}_n) Y$ in the integral in (1.1), we find by Lemma 4.2(f)

$$\int \exp(-nF_n(Y))\, dP_n(\sqrt{n} Y) = \det(I + \sqrt{n}\mathscr{J}_n) \int \exp(-n\Delta_n(Y))\, dP_n(Y).$$

By Lemma 4.3(c), we obtain (1.1) once we show

$$\frac{1}{n} \ln \int \exp(-n\Delta_n(Y))\, dP_n(Y) \to -G(Y^*).$$

We do this by proving the lower bound

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int \exp(-n\Delta_n(Y))\, dP_n(Y) \geq -G(Y^*) \tag{4.19}$$

and the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \ln \int \exp(-n\Delta_n(Y))\, dP_n(Y) \leq -G(Y^*). \tag{4.20}$$

PROOF OF LOWER BOUND (4.19). We work with the sets $\mathscr{K}_n(\theta, C_2)$ defined in (4.10), setting $\bar{Y} = Y^*$. Assume $0 < \theta < 1$ and let $\{(Y^*)^{(n)}\}$ be the sequence of elements constructed in Lemma 4.4(c) for $Y = Y^*$. Given arbitrary $\varepsilon > 0$, we prove below that there exists $\bar{\theta} = \bar{\theta}(\varepsilon) \in (0, 1)$ such that for $Y \in \mathscr{K}_n(\bar{\theta}, \bar{C}_2)$, n sufficiently large,

$$| G_n(\mathscr{J}_n Y) - G_n((Y^*)^{(n)}) | < \varepsilon. \tag{4.21}$$

$\bar{C}_2 = \bar{C}_2(\bar{\theta})$ is picked so that (see Lemma 4.2(g))

$$\liminf_{n\to\infty} P_n(\mathcal{K}_n(\bar{\theta}, \bar{C}_2)) > 0. \tag{4.22}$$

We now prove that there exists $C_3 = C_3(\bar{\theta})$ such that for all sufficiently large n

$$\Delta_n(Y) \le G_n((Y^*)^{(n)}) + \varepsilon + \frac{C_3}{n^{\gamma/2}}, \qquad \text{all } Y \in \mathcal{K}_n(\bar{\theta}, \bar{C}_2), \tag{4.23}$$

where γ is the number in Hypothesis 3.3. Then

$$\begin{aligned} \int \exp(-n\Delta_n)\, dP_n &\ge \int_{\mathcal{K}_n(\bar{\theta},\bar{C}_2)} \exp(-n\Delta_n)\, dP_n \\ &\ge \exp\left[-n\left(G_n((Y^*)^{(n)}) + \varepsilon + \frac{C_3}{n^{\gamma/2}}\right)\right] P_n(\mathcal{K}_n(\bar{\theta}, \bar{C}_2)) \end{aligned} \tag{4.24}$$

and (4.19) follows from $G_n((Y^*)^{(n)}) \to G(Y^*)$ (Lemma 4.5(b)) and (4.22). To prove (4.23), we have by (4.16)(i) for any $Y \in \mathcal{K}_n(\bar{\theta}, \bar{C}_2)$

$$\max\left(\frac{\|Y\|}{\sqrt{n}}, \left\|\mathcal{J}_n Y + \frac{Y}{\sqrt{n}}\right\|\right) \le \|\mathcal{J}_n Y\| + \|Y\| \le \tilde{R} := (K+1)\bar{C}_2.$$

Hence, if n is sufficiently large, then by (4.21) and (3.2),

$$\begin{aligned} \Delta_n(Y) &= G_n(\mathcal{J}_n Y) + F_n\left(\frac{Y}{\sqrt{n}} + \mathcal{J}_n Y\right) - F_n(\mathcal{J}_n Y) + \frac{1}{\sqrt{n}}\|Y\|_2^2 \\ &\le G_n((Y^*)^{(n)}) + \varepsilon + \frac{C(\tilde{R})}{n^{\gamma/2}}\|Y\|_2^\gamma + \frac{1}{\sqrt{n}}\|Y\|_2^2 \\ &\le G_n((Y^*)^{(n)}) + \varepsilon + \frac{C_3}{n^{\gamma/2}}, \end{aligned}$$

where $C_3 := C(\tilde{R})\bar{C}_2^\gamma + \bar{C}_2^2$. This proves (4.23).

We now prove (4.21). Define $R := \sqrt{K}(\|\mathcal{J}^{-1/2}Y^*\|_2 + 1)$, where $\sqrt{K}$ is the number in (4.16)(iii). Then by (4.16)(iii), for any $Y \in \mathcal{K}_n(\theta, C_2)$

$$\|\mathcal{J}_n Y\| \le \sqrt{K}\|\sqrt{\mathcal{J}_n} Y\|_2 \le \sqrt{K}(\|\mathcal{J}^{-1/2}Y^*\|_2 + 1) = R.$$

Also, $\|Y^*\| \le \sqrt{K}\|\mathcal{J}^{-1/2}Y^*\|_2 < R$. Hence if n is sufficiently large, then by (3.2), (4.16)(iii)

$$\begin{aligned} |G_n(\mathcal{J}_n Y) - G_n((Y^*)^{(n)})| &\le |I_n(\mathcal{J}_n Y) - I_n((Y^*)^{(n)})| + |F_n(\mathcal{J}_n Y) - F_n((Y^*)^{(n)})| \\ &\le \frac{C_4}{2}\|\sqrt{\mathcal{J}_n} Y - \mathcal{J}_n^{-1/2}(Y^*)^{(n)}\|_2 + C(R)\|\mathcal{J}_n Y - (Y^*)^{(n)}\|_2^\gamma \\ &\le \left(\frac{C_4}{2} + C(R)\sqrt{K}^\gamma\right)\|\sqrt{\mathcal{J}_n} Y - \mathcal{J}_n^{-1/2}(Y^*)^{(n)})\|_2^\gamma, \end{aligned}$$

where

$$C_4 := \sup\{\|\sqrt{\mathcal{J}_n} Y\|_2 + \|\mathcal{J}_n^{-1/2}(Y^*)^{(n)}\|_2 : n,\ Y \in \mathcal{K}_n(\theta, C_2)\}.$$

But $\mathcal{J}_n^{-1/2}(Y^*)^{(n)} = \tau_n \mathcal{J}^{-1/2}Y^*$, so that $C_4 \le 2\|\mathcal{J}^{-1/2}Y^*\| + 1 < \infty$. For $Y \in \mathcal{K}_n(\theta, C_2)$

$$|G_n(\mathcal{J}_n Y) - G_n((Y^*)^{(n)})| \le \left(\frac{C_4}{2} + C(R)\sqrt{K}^\gamma\right)\theta^\gamma.$$

This can be made less than ε by picking $\theta \in (0, 1)$ sufficiently small. This proves (4.21) and completes the proof of the lower bound (4.19).

PROOF OF UPPER BOUND (4.20). For $R > 0$, define the subsets of $C[0, 1]$ which partition $C[0, 1]$:

$$\mathscr{V}_n^{(1)} := \{Y: \|\mathscr{J}_n Y\| \le R, \|Y\| \le R\sqrt{n}\},$$
$$(4.25) \qquad \mathscr{V}_n^{(2)} := \{Y: \|Y\| > R\sqrt{n}\},$$
$$\mathscr{V}_n^{(3)} := \{Y: \|\mathscr{J}_n Y\| > R\}.$$

We prove that there exists a constant $C_5 > 0$ so that for all sufficiently large R and n

$$(4.26) \qquad \int_{\mathscr{V}_n^{(i)}} \exp(-n\Delta_n)\, dP_n \le \begin{cases} \exp(-nG_n(Y_n^*))\exp\left(C_5 n^{\frac{4-3\gamma}{4-2\gamma}}\right), & i = 1, \\ \exp(-nG(Y^*)), & i = 2, 3, \end{cases}$$

where γ is the number in Hypothesis 3.3. This yields (4.20) since $G_n(Y_n^*) \to G(Y^*)$ (Lemma 4.5(e)) and $(4 - 3\gamma)/(4 - 2\gamma) < 1$.

Upper bound, region 1. For any $Y \in \mathscr{V}_n^{(1)}$, $\max_n\{\|Y/\sqrt{n}\|, \|Y/\sqrt{n} + \mathscr{J}_n Y\|\} \le 2R$. Hence, for sufficiently large n, by the minimizing property of Y_n^* and (3.2)

$$(4.27) \qquad \begin{aligned} \Delta_n(Y) &= G_n(\mathscr{J}_n Y) + F_n\left(\frac{Y}{\sqrt{n}} + \mathscr{J}_n Y\right) - F_n(\mathscr{J}_n Y) + \frac{\|Y\|_2^2}{\sqrt{n}} \\ &\ge G_n(Y_n^*) - \frac{C(2R)}{n^{\gamma/2}} \|Y\|_2^\gamma + \frac{\|Y\|_2^2}{\sqrt{n}}. \end{aligned}$$

For any numbers $r > 0$, $s > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have the inequality

$$r^p - sr \ge -\frac{p}{q}\left(\frac{s}{p}\right)^q.$$

Let $r := \|Y\|_2^\gamma$, $s := C(2R)n^{(1-\gamma)/2}$, $p := 2/\gamma$, $q := 2/(2 - \gamma)$. Then

$$\begin{aligned} \frac{\|Y\|_2^2}{\sqrt{n}} - \frac{C(2R)}{n^{\gamma/2}} \|Y\|_2^\gamma &= \frac{1}{\sqrt{n}} (\|Y\|_2^2 - C(2R)n^{(1-\gamma)/2} \|Y\|_2^\gamma) \\ &\ge -C_5 n^{-\gamma/2(2-\gamma)}, \end{aligned}$$

where C_5 depends on γ. Inserting this into (4.27), we obtain (4.26) for $i = 1$ and all sufficiently large n.

Upper bound, region 2. For any $Y \in C[0, 1]$ and n sufficiently large, (3.2) implies

$$\begin{aligned} \Delta_n(Y) &= F_n\left(\frac{Y}{\sqrt{n}} + \mathscr{J}_n Y\right) + \frac{1}{\sqrt{n}} \|Y\|_2^2 + \frac{1}{2} (\mathscr{J}_n Y, Y) \\ &\ge -\nu \|Y/\sqrt{n} + \mathscr{J}_n Y\|_2^2 - C_1 + \frac{1}{\sqrt{n}} \|Y\|_2^2 + \frac{1}{2} \|\sqrt{\mathscr{J}_n} Y\|_2^2. \end{aligned}$$

But

$$\begin{aligned} \|Y/\sqrt{n} + \mathscr{J}_n Y\|_2^2 &= (\mathscr{J}_n Y, \mathscr{J}_n Y) + \frac{2}{\sqrt{n}} \|\sqrt{\mathscr{J}_n} Y\|_2^2 + \frac{1}{n} \|Y\|_2^2 \\ &\le \left(\|\mathscr{J}_n\|_2 + \frac{2}{\sqrt{n}}\right) \|\sqrt{\mathscr{J}_n} Y\|_2^2 + \frac{1}{n} \|Y\|_2^2. \end{aligned}$$

Hence $$\Delta_n(Y) \ge a_n \|\sqrt{\mathscr{J}_n} Y\|_2^2 + b_n \|Y\|_2^2 - C_1,$$

where

$$a_n := \frac{1}{2} - \nu \|\mathscr{J}_n\|_2 - \frac{2\nu}{\sqrt{n}}, \qquad b_n := \frac{1}{\sqrt{n}} - \frac{\nu}{n}. \tag{4.28}$$

For all sufficiently large n, we have $a_n \geq 0$, $b_n \geq 0$, and so $\Delta_n(Y) \geq -C_1$. Thus by (4.5)

$$\int_{\mathscr{V}_n^{(2)}} \exp(-n\Delta_n)\, dP_n \leq \exp(nC_1)P_n(\mathscr{V}_n^{(2)}) \leq C \exp(-n(\beta R^2 - C_1)).$$

If we pick $\bar{R}$ so that $\beta\bar{R}^2 - C_1(\nu) \geq G(Y^*)$, then (4.26) for $i = 2$ holds for all $R \geq \bar{R}$ and all sufficiently large n.

Upper bound, region 3. For any $Y \in \mathscr{V}_n^{(3)}$, (4.16)(iii) implies $\|\sqrt{\mathscr{J}_n}Y\|_2^2 \geq K^{-1}\|\mathscr{J}_nY\|^2 \geq K^{-1}R^2$. In (4.28), there exists $\delta > 0$ so that for all sufficiently large n, $a_n \geq \delta$, $b_n \geq 0$. Thus on $\mathscr{V}_n^{(3)}$, $\Delta_n(Y) \geq \delta K^{-1}R^2 - C_1$, which can be made to exceed $G(Y^*)$ by taking R large enough. Thus, (4.26) for $i = 3$ holds for all sufficiently large R and n.

This completes the proof of the upper bound (4.20) and thus of Theorem 1.1.

V. Proof of Theorem 1.3. We first state Lemma 5.1, then show how Theorem 1.3 follows directly from it. Afterwards we prove the lemma. The proof of the lemma depends in part upon the bound (1.12), which we deduce from (1.1).

LEMMA 5.1. *Define the measures* $\{\Lambda_n\}$ *on* $C[0, 1]$ *by*

$$d\Lambda_n(Y) := \exp(-nF_n(Y))\, dP_n(\sqrt{n}Y). \tag{5.1}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \ln \Lambda_n\{C[0, 1]\} = -\inf_{Y\in C[0,1]} G(Y) \tag{5.2}$$

while for any closed subset $\mathscr{A}$ *of* $C[0, 1]$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \Lambda_n\{\mathscr{A}\} \leq -\inf_{Y\in\mathscr{A}} G(Y). \tag{5.3}$$

PROOF OF THEOREM 1.3. Let $\mathscr{A} \subseteq C[0, 1]$ be closed. Then by (5.2)–(5.3)

$$\limsup_{n\to\infty} \frac{1}{n} \ln Q_n\{\mathscr{A}\} = \limsup_{n\to\infty} \frac{1}{n} \ln\left(\frac{\Lambda_n\{\mathscr{A}\}}{\Lambda_n\{C[0, 1]\}}\right)$$
$$\leq -\inf_{Y\in\mathscr{A}} [G(Y)] + \inf_{Y\in C[0,1]} [G(Y)].$$

Thus if $\mathscr{A}$ also satisfies (1.10), then we can find a positive number N so that $Q_n(\mathscr{A}) < e^{-n\delta}$ for all $n \geq N$. This is (1.11). □

PROOF OF LEMMA 5.1. The limit (5.2) is exactly (1.1). In Varadhan (1966) Section 3, (5.3) is shown to hold if (1.12) is valid, if the F, $\{F_n\}$ satisfy Hypothesis 3.4, and if (1.14) holds. We save the proof of (1.12) for last. We now prove (1.14). For all sufficiently large n and L, (3.2) and (4.4) imply

$$P_n(\sqrt{n}\{Y: -F_n(Y) \geq L\}) \leq P_n\left\{Y: \|Y\|^2 \geq \frac{nL}{2\nu}\right\} \leq C \exp\left[-n\beta\left(\frac{L}{2\nu}\right)\right]. \tag{5.4}$$

We pick $q > 1$ so that $q\nu < 1/(2\|\mathscr{J}\|_2)$ and let p satisfy $p^{-1} + q^{-1} = 1$. The functionals qF, $\{qF_n\}$ then also satisfy Hypotheses 3.2–3.4. We have by Theorem 1.1 (applied to these functionals) and by (5.4) that for all sufficiently large n the integral in (1.14) is bounded above by

$$[P_n(\sqrt{n}\{Y: -F_n(Y) \geq L\})]^{1/p}\left\{\int e^{-nqF_n(Y)} dP_n(\sqrt{n}Y)\right\}^{1/q} \leq C \exp\left[\frac{-n\beta}{p}\left(\frac{L}{2\nu}\right)\right]\exp(nD),$$

where D is some real number. Now (1.14) follows.

To prove (1.12), we need another lemma.

LEMMA 5.2. *Given $\mathscr{A} \subseteq C[0, 1]$ closed and $\delta > 0$, define the subset of $C[0, 1]$*

$$\mathscr{A}^{\delta} := \{Y : d(Y, \mathscr{A}) > \delta\},$$

where $d(Y, \mathscr{A}) := \inf_{X \in \mathscr{A}} \|Y - X\|_2$. Define the functional

$$\Phi_{\delta}(Y) := \frac{d(Y, \mathscr{A})}{d(Y, \mathscr{A}) + d(Y, \mathscr{A}^{\delta})}. \tag{5.5}$$

Then for any number $b \geq 0$

$$\lim_{n \to \infty} \frac{1}{n} \ln \int \exp[-nb\Phi_{\delta}(Y)]\, dP_n(\sqrt{n}Y) = -\inf_{Y \in C[0,1]}[b\Phi_{\delta}(Y) + I(Y)]. \tag{5.6}$$

PROOF. We show that

$$\begin{aligned} &\text{(a) } 0 \leq \Phi_{\delta}(Y) \leq 1 \quad \text{for all } Y \in C[0, 1],\\ &\text{(b) } |\Phi_{\delta}(X) - \Phi_{\delta}(Y)| \leq \frac{2}{\delta}\|X - Y\|_2 \quad \text{for all } X, Y \in C[0, 1],\\ &\text{(c) if } Y \in C[0, 1], \{Y_n\} \subseteq C[0, 1], \quad \text{and } Y_n \to Y, \quad \text{then } \Phi_{\delta}(Y_n) \to \Phi_{\delta}(Y). \end{aligned} \tag{5.7}$$

Then for $b \geq 0$, $F_1 = F_2 = \cdots = F := b\Phi_{\delta}$ satisfy Hypotheses 3.2–3.4 and so (5.6) follows from (1.1). Property (5.7)(a) is obvious and (5.7)(c) follows from (5.7)(b) and $\|X - Y\|_2 \leq \|X - Y\|$. To prove (5.7)(b), we have

$$\Phi_{\delta}(X) - \Phi_{\delta}(Y) = \frac{[d(X, \mathscr{A}) - d(Y, \mathscr{A})]d(Y, \mathscr{A}^{\delta}) + [d(Y, \mathscr{A}^{\delta}) - d(X, \mathscr{A}^{\delta})]d(Y, \mathscr{A})}{[d(X, \mathscr{A}) + d(X, \mathscr{A}^{\delta})][d(Y, \mathscr{A}) + d(Y, \mathscr{A}^{\delta})]}.$$

Thus,

$$|\Phi_{\delta}(X) - \Phi_{\delta}(Y)| \leq \frac{1}{\delta}|d(X, \mathscr{A}) - d(Y, \mathscr{A})| + \frac{1}{\delta}|d(X, \mathscr{A}^{\delta}) - d(Y - \mathscr{A}^{\delta})| \leq \frac{2}{\delta}\|X - Y\|_2.$$

□

We are now ready to prove (1.12). We must show that for $\mathscr{A} \subseteq C[0, 1]$ closed

$$\limsup_{n \to \infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{A}) \leq -\inf_{Y \in \mathscr{A}}[I(Y)]. \tag{5.8}$$

We note that

$$\Phi_{\delta} \restriction \mathscr{A} \equiv 0, \quad \Phi_{\delta} \restriction \mathscr{A}^{\delta} \equiv 1, \quad 0 \leq \Phi_{\delta} \leq 1 \quad \text{on} \quad C[0, 1]. \tag{5.9}$$

This and (5.6) imply that for any number $b \geq 0$

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{A}) &\leq \limsup_{n \to \infty} \frac{1}{n} \ln \int_{\mathscr{A}} \exp[-nb\Phi_{\delta}(Y)]\, dP_n(\sqrt{n}Y)\\ &\leq \lim_{n \to \infty} \frac{1}{n} \ln \int \exp(-nb\Phi_{\delta}(Y))\, dP_n(\sqrt{n}Y)\\ &= -\inf_{Y \in C[0,1]}[b\Phi_{\delta}(Y) + I(Y)]. \end{aligned} \tag{5.10}$$

Since the left-hand term in (5.10) is independent of b, δ, we find

$$\limsup_{n \to \infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{A}) \leq -\limsup_{\delta \downarrow 0}(\sup_{b \geq 0} \inf_{Y \in C[0,1]}[b\Phi_{\delta}(Y) + I(Y)]). \tag{5.11}$$

Define the subset $\mathscr{A}_{\delta} := \{Y : d(Y, \mathscr{A}) \leq \delta\}$ and the number $r := \lim_{\delta \downarrow 0}(\inf_{Y \in \mathscr{A}_{\delta}}[I(Y)])$; r is well-defined ($0 \leq r \leq \infty$) since the $\{\mathscr{A}_{\delta}\}$ are decreasing as $\delta \downarrow 0$. Together with (5.11), the facts

(5.12) $$\sup_{b\ge 0} \inf_{Y\in C[0,1]}[b\Phi_\delta(Y) + I(Y)] \ge \inf_{Y\in\mathscr{A}_\delta}[I(Y)]$$

and

(5.13) $$r = \inf_{Y\in\mathscr{A}}[I(Y)]$$

will yield (5.8). We first prove (5.12). By (5.9), we have for any $b \ge 0$

$$\inf_{Y\in\mathscr{A}_\delta}\{b\Phi_\delta(Y) + I(Y)] \ge \inf_{Y\in\mathscr{A}_\delta}[I(Y)],$$
$$\inf_{Y\in\mathscr{A}^\delta}[b\Phi_\delta(Y) + I(Y)] = b + \inf_{Y\in\mathscr{A}^\delta}[I(Y)].$$

Thus,

$$\begin{aligned}\sup_{b\ge 0} &\inf_{Y\in C[0,1]}[b\Phi_\delta(Y) + I(Y)]\\ &= \sup_{b\ge 0}\min\{\inf_{Y\in\mathscr{A}_\delta}[b\Phi_\delta(Y) + I(Y)], \quad \inf_{Y\in\mathscr{A}^\delta}[b\Phi_\delta(Y) + I(Y)]\}\\ &\ge \min\{\inf_{Y\in\mathscr{A}_\delta}[I(Y)], \sup_{b\ge 0}(b) + \inf_{Y\in\mathscr{A}^\delta}[I(Y)]\} = \inf_{Y\in\mathscr{A}_\delta}[I(Y)].\end{aligned}$$

This is (5.12). To prove (5.13), we note that since $\mathscr{A} \subset \mathscr{A}_\delta$ for each $\delta > 0$,

(5.14) $$\inf_{Y\in\mathscr{A}}[I(Y)] \ge r.$$

Thus, if r $+\infty$, (5.13) is automatic. Assuming $r < \infty$, we are done once we prove

(5.15) $$r \ge \inf_{Y\in\mathscr{A}}[I(Y)].$$

For each $\delta > 0$, define $i(\delta) := \inf_{Y\in\mathscr{A}_\delta}[I(Y)]$; we have $i(\delta)\uparrow r$ as $\delta\downarrow 0$.

Since $r < \infty$ by assumption, we see that $0 \le i(\delta) < \infty$. Now clearly

$$\inf_{Y\in\mathscr{A}_\delta}[I(Y)] = \inf_{Y\in\tilde{\mathscr{A}}_\delta}[I(Y)], \qquad \text{where} \quad \tilde{\mathscr{A}}_\delta := \mathscr{A}_\delta \cap \{Y: I(Y) \le i(\delta) + 1\}. \quad \square$$

Since $\mathscr{A}_\delta$ is closed in $C[0, 1]$, $\tilde{\mathscr{A}}_\delta$ is compact by Lemma 4.4(b). Since I is lower semicontinuous, there exists $Y_\delta \in \tilde{\mathscr{A}}_\delta$ so that $I(Y_\delta) = i(\delta)$. Let $\{\delta_i\}$ be any sequence tending to zero. Then

(5.16) $$\lim_{i\to\infty} i(\delta_i) = \lim_{i\to\infty} I(Y_{\delta_i}) = r.$$

By Lemma 4.4(b), $\{Y_{\delta_i})$ is conditionally compact, so there exists a subsequence $\{Y_{\delta_{i'}}\}$ and an element $\bar{Y} \in \cap_{i'} \tilde{\mathscr{A}}_{\delta_{i'}} \subseteq \mathscr{A}$ so that $Y_{\delta_{i'}} \to \bar{Y}$. The lower semicontinuity of I and (5.16) imply $r = \lim_{i'\to\infty} I(Y_{\delta_{i'}}) \ge I(\bar{Y}) \ge \inf_{Y\in\mathscr{A}}[I(Y)]$, which is (5.15).

APPENDIX

Laplace's Method and Related Results for a Separable Real Hilbert Space

We prove analogues of Theorems 1.1, 1.3, and 1.4 for a separable real Hilbert space $\mathscr{H}$. We denote the norm of $\mathscr{H}$ by $\|-\|$ and the inner product of $\mathscr{H}$ by $(-, -)$. Given elements Y, $\{Y_n\}$ of $\mathscr{H}$, we write $Y_n \rightharpoonup Y$ if $(Y_n, X) \to (Y, X)$ for all $X \in \mathscr{H}$ and $Y_n \to Y$ if $\|Y - Y_n\| \to 0$. Let A, $\{A_n\}$ be bounded linear mappings from $\mathscr{H}$ to $\mathscr{H}$. We write $\|A\|$ to denote the operator norm of A. We write $A_n \rightharpoonup A$ if $A_n Y \rightharpoonup AY$ for all $Y \in \mathscr{H}$, $A_n \to A$ if $A_n Y \to AY$ for all $Y \in \mathscr{H}$, and $A_n \twoheadrightarrow A$ if $\|A - A_n\| \to 0$.

Let P, $\{P_n\}$ be mean zero Gaussian measures on $\mathscr{H}$ with covariance operators $\mathscr{J}$, $\{\mathscr{J}_n\}$. For any Y_1, $Y_2 \in \mathscr{H}$, we have [Gihman-Skorohod (1974) page 341]

(A.1) $$(\mathscr{J}Y_1, Y_2) = \int (Y_1, Y)(Y_2, Y)\, dP(Y), \; (\mathscr{J}_n Y_1, Y_2) = \int (Y_1, Y)(Y_2, Y)\, dP_n(Y).$$

In this Appendix, all integrals with respect to P, $\{P_n\}$ are over $\mathscr{H}$ unless otherwise noted. $\mathscr{J}$ and each $\mathscr{J}_n$ are positive, symmetric, trace class operators. We denote by τ, τ_n the orthogonal projections of $\mathscr{H}$ onto the subspaces $\{Y: \mathscr{J}Y \ne 0\}$, $\{Y: \mathscr{J}_n Y \ne 0\}$; by $\sqrt{\mathscr{J}}$, $\{\sqrt{\mathscr{J}_n}\}$ the unique symmetric, positive square roots of $\mathscr{J}$, $\{\mathscr{J}_n\}$; by $\mathscr{D}$, $\{\mathscr{D}_n\}$ the ranges of these operators. $\sqrt{\mathscr{J}}$ is invertible on $\mathscr{D}$ with inverse $\mathscr{J}^{-1/2}$; each $\sqrt{\mathscr{J}_n}$ is invertible on $\mathscr{D}_n$ with inverse $\mathscr{J}_n^{-1/2}$. We define the entropy functionals I, $\{I_n\}$ of P, $\{P_n\}$ by

$$I(Y) := \begin{cases} \frac{1}{2}\|\mathscr{J}^{-1/2}Y\|^2 & \text{if } Y \in \mathscr{D}, \\ +\infty & \text{if } Y \in \mathscr{H}\backslash\mathscr{D}, \end{cases}$$

$$I_n(Y) := \begin{cases} \frac{1}{2}\|\mathscr{J}_n^{-1/2}Y\|^2 & \text{if } Y \in \mathscr{D}_n, \\ +\infty & \text{if } Y \in \mathscr{H}\backslash\mathscr{D}_n. \end{cases}$$

The entropy functional of a Gaussian measure on a general Banach space is defined in Donsker-Varadhan (1976) Theorem 6.2. One may check that for a Hilbert space their definition reduces to the above. See also Freidlin (1972) and Wentzell (1972).

We now state the hypotheses under which the theorems of this section will be proved.

HYPOTHESIS A.1. *P, $\{P_n\}$ are mean zero Gaussian measures on $\mathscr{H}$ such that $P_n \Rightarrow P$.*

HYPOTHESIS A.2. *F, $\{F_n\}$ are real-valued functionals on $\mathscr{H}$. There exist numbers ν and C_1 satisfying $0 < \nu < 1/(2\|\mathscr{J}\|)$ and $C_1 > 0$ such that for any $Y \in \mathscr{H}$ and all sufficiently large n*

$$F(Y) \geq -\nu\|Y\|^2 - C_1, \qquad F_n(Y) \geq -\nu\|Y\|^2 - C_1.$$

HYPOTHESIS A.3. *There exists $\gamma \in (0, 1]$ and, for each $R > 0$, there exists $C(R) > 0$ such that if $\|X\| \leq R$, $\|Y\| \leq R$, then for all sufficiently large n*

$$|F(X) - F(Y)| \leq C(R)\|X - Y\|^\gamma, \qquad |F_n(X) - F_n(Y)| \leq C(R)\|X - Y\|^\gamma.$$

HYPOTHESIS A.4. *Given $Y \in \mathscr{D}$ and $\{Y_n\}$ a sequence such that $Y_n \to Y$, then $F_n(Y_n) \to F(Y)$.*

The following three theorems correspond to Theorem 1.1, Theorem 1.4, and Theorem 1.5, respectively.

THEOREM A.5. *Assume that P, $\{P_n\}$, F, $\{F_n\}$ satisfy Hypotheses A.1–1.4. Then*

$$\text{(A.2)} \qquad \lim_{n\to\infty} \frac{1}{n} \ln \int \exp[-nF_n(Y)]\, dP_n(\sqrt{n}Y) = -\inf_{Y\in\mathscr{H}}[F(Y) + I(Y)].$$

Further, $\inf[F + I]$ *over $\mathscr{H}$ is finite and is attained at some point in $\mathscr{H}$.*

THEOREM A.6. *Let P, $\{P_n\}$, F, $\{F_n\}$ satisfy Hypotheses* A.1–A.4. *Define probability measures Q_n on $\mathscr{H}$ as in* (1.9). *Assume that $\mathscr{A}$ is a closed subset of $\mathscr{H}$ such that for some $\delta > 0$*

$$\inf_{Y\in\mathscr{A}}[F(Y) + I(Y)] - \min_{Y\in\mathscr{H}}[F(Y) + I(Y)] > \delta.$$

Then for all sufficiently large n, $Q_n(\mathscr{A}) \leq e^{-n\delta}$.

THEOREM A.7. *Assume that P, $\{P_n\}$ satisfy Hypothesis* A.1. *Then for any closed subset $\mathscr{A}$ of $\mathscr{H}$*

$$\limsup_{n\to\infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{A}) \leq -\inf_{Y\in\mathscr{A}} I(Y)$$

and for any open subset $\mathscr{B}$ of $\mathscr{H}$

$$\liminf_{n\to\infty} \frac{1}{n} \ln P_n(\sqrt{n}\mathscr{B}) \geq -\inf_{Y\in\mathscr{B}} I(Y).$$

Also, (1.1) *holds if F, $\{F_n\}$ satisfy only Hypotheses* A.4 *and the F_n satisfy* (1.14). *The second assertion in Theorem* A.5 *follows if F is lower semicontinuous on H. Finally, we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln P_n\{Y: \|Y\| \geq \sqrt{n}\} = -\frac{1}{2\|\mathscr{J}\|}.$$

Much of the work needed to prove these theorems has already been done. Except for changes in notation and in the proofs of several lemmas, these theorems are proved in exactly the same way as their analogues in Section I. The changes in notation are obvious: replace $C[0, 1]$ and L^2 by $\mathscr{H}$ and replace the supremum norm on $C[0, 1]$ and the L^2-norm $\| - \|_2$ by $\| - \|$, the norm on $\mathscr{H}$. We now state those facts needed to prove Theorems A.5–A.7 for which the proofs given earlier in this paper do not immediately generalize.

LEMMA A.8.

(a) *There exist positive constants β and c so that for all n*

$$P_n\{Y: \|Y\| > a\} > Ce^{-\beta a^2}, \qquad \textit{all } a > 0. \tag{A.3}$$

(b) $\frac{1}{n} \ln \det(I + \sqrt{n}\mathscr{J}_n) \to 0$.

(c) $\mathscr{J}_n \twoheadrightarrow \mathscr{J}$.

(d) $\sqrt{\mathscr{J}_n} \twoheadrightarrow \sqrt{\mathscr{J}}$.

(e) *Let $\{Y_n\}$ be a sequence in $\mathscr{H}$. If $\sup I_n(Y_n) < \infty$, then $\{Y_n\}$ has a convergent subsequence. In particular, for any $L \in (0, \infty)$, the set $\{Y: I(Y) \leq L\}$ is compact.*

(f) *For all $Y \in \mathscr{H}$ and A equal to $\mathscr{J}$ or any $\mathscr{J}_n$, there exists a constant $K > 0$ so that* $\|AY\| \leq K\|Y\|$, $\|\sqrt{A}\, Y\| \leq \sqrt{K}\|Y\|$, $\|AY\| \leq \sqrt{K}\|\sqrt{A}\, Y\|$.

PROOF OF LEMMA A.8.

(a) Let Q be any mean zero Gaussian measure on $\mathscr{H}$. We prove below that if $q > ½$ and $S > 0$ have the property that

$$Q\{Y: \|Y\| \leq S\} \geq q, \tag{A.4}$$

then there exists $\beta > 0$ such that for all $a > S$

$$Q\{Y: \|Y\| > a\} \leq e^{-\beta a^2}. \tag{A.5}$$

Using this fact, one completes the proof of (a) exactly like the proof of Lemma 4.2(a). Since $\mathscr{H}$ is separable, there exists a countable set $\{e_n\}$ in $\mathscr{H}$ such that for any $Y \in \mathscr{H}$

$$\|Y\| = \sup_n (Y, e_n).$$

Hence the proof of Theorem 1.9 in Marcus-Shepp (1972) can be used to show that (A.4) implies (A.5).

(b) We define

$$K := \sup_n \int \|Y\|^2 \, dP_n(Y); \tag{A.6}$$

K is finite because of (A.3). By (A.1) $\mathrm{Tr}\mathscr{J}_n \leq K$ and so this part is proved like Lemma 4.3(c).

(c) We prove that

$$\lim_{n\to\infty} \|J - J_n\| = \lim \sup_{n\to\infty \|X\| \leq 1} |((J - J_n)X, X)| = 0. \tag{A.7}$$

Let $\varepsilon > 0$ be given. Since $P_n \Rightarrow P$, Prohorov's theorem [Billingsley (1968) Theorem 6.2] implies that there exists a compact subset K_ε of $\mathscr{H}$ such that $P(K_\varepsilon^c) \leq \varepsilon$, $P_n(K_\varepsilon^c) \leq \varepsilon$ for all n. By (A.1),

$$\text{(A.8)} \qquad \begin{aligned} |((J - J_n)X, X)| \le & \left| \int_{K_\varepsilon} (X, Y)^2 \, dP(Y) - \int_{K_\varepsilon} (X, Y)^2 \, dP_n(Y) \right| \\ & + \int_{K_\varepsilon^c} (X, Y)^2 \, dP(Y) + \int_{K_\varepsilon^c} (X, Y)^2 \, dP_n(Y). \end{aligned}$$

By (A.3) $\sup_n \int \| Y \|^4 \, dP_n(Y) < \infty$. Hence uniformly in n

$$\sup_{\|X\| \le 1} \int_{K_\varepsilon^c} (X, Y)^2 \, dP_n(Y) \le [P_n(K_\varepsilon^c)]^{1/2} \left[\int \| Y \|^2 \, dP_n(Y) \right]^{1/2} = O(\varepsilon^{1/2});$$

the same bound holds for $\sup_{\|X\| \le 1} \int_{K_\varepsilon^c} (X, Y)^2 \, dP(Y)$. We prove that there exists $N > 0$ so that for all $n \ge N$

$$\text{(A.9)} \qquad \sup_{\|X\| \le 1} \left| \int_{K_\varepsilon} (X, Y)^2 \, dP(Y) - \int_{K_\varepsilon} (X, Y)^2 \, dP_n(Y) \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (A.7) follows. To prove (A.9), let $\mathscr{E}$ be the set of all functionals f on K_ε of the form $f(Y) = (X, Y)^2$ for some X satisfying $\|X\| \le 1$. We prove below that $\mathscr{E}$ is compact in the topology of uniform convergence on K_ε. Because of the compactness, we can find elements $g_1, \cdots, g_M \in \mathscr{E}$, some $M < \infty$, with the property that for each $f \in \mathscr{E}$ there exists an $i \in \{1, \cdots, M\}$ such that

$$\text{(A.10)} \qquad \sup_{Y \in K_\varepsilon} |f(Y) - g_i(Y)| < \varepsilon/2.$$

Since $P_n \Rightarrow P$, there exists $N > 0$ such that for all $i \in \{1, \cdots, M\}$, all $n \ge N$

$$\text{(A.11)} \qquad \left| \int_{K_\varepsilon} g_i(Y) \, dP(Y) - \int_{K_\varepsilon} g_i(Y) \, dP_n(Y) \right| < \frac{\varepsilon}{2}.$$

Together, (A.10)–(A.11) imply (A.9). We now prove the compactness of $\mathscr{E}$. Let $\{f_{n_1}\}$ be an arbitrary sequence in $\mathscr{E}$. Then $f_{n_1}(Y) = (X_{n_1}, Y)^2$, where each $\|X_{n_1}\| \le 1$. By Berger (1977) (1.3.12)(iii), there exists a subsequence $\{X_{n_2}\}$ and an element $X \in \mathscr{H}$ so that $X_{n_2} \rightharpoonup X$. Defining $f(X) := (X, Y)^2$, we prove $f_{n_2} \to f$, which will complete the proof. If the latter were false, then there would exist $\beta > 0$, an infinite subsubsequence $\{n_3\}$, and elements $\{Y_{n_3}\}$ of K_ε such that for all n_3

$$\text{(A.12)} \qquad |(X_{n_3}, Y_{n_3})^2 - (X, Y_{n_3})^2| \ge \beta.$$

Since K_ε is compact, we can assume that the $\{Y_{n_3}\}$ converge strongly to some element $\bar{Y}$ of K_ε. But then both inner products in (A.12) tend to (X, Y) and so (A.12) cannot hold.

(d) This is implied by (c).

(e) Define $X_n := \mathscr{I}_n^{-1/2} Y_n$. Then $D := \sup_n \|X_n\|^2 = 2 \sup_n I_n(Y_n) < \infty$. Hence there exists a subsequence $\{X_{n'}\}$ and an element $X \in H$ such that $X_{n'} \rightharpoonup X$. We show $Y_{n'} = \sqrt{\mathscr{I}_{n'}} X_{n'} \to \sqrt{\mathscr{I}} X$, which completes the proof. We have

$$\begin{aligned} \| \sqrt{\mathscr{I}} X - \sqrt{\mathscr{I}_{n'}} X_{n'} \| & \le \| (\sqrt{\mathscr{I}} - \sqrt{\mathscr{I}_{n'}}) X_{n'} \| + \| \sqrt{\mathscr{I}} X_{n'} - \sqrt{\mathscr{I}} X \| \\ & \le \sqrt{D} \| \sqrt{\mathscr{I}} - \sqrt{\mathscr{I}_{n'}} \| + \| \sqrt{\mathscr{I}} X_{n'} - \sqrt{\mathscr{I}} X \|. \end{aligned}$$

The first term tends to zero by (d) and the second term tends to zero because $X_{n'} \rightharpoonup X$ and $\sqrt{\mathscr{I}}$ is compact.

(f) By (A.1) we have that $\sup_n \|\mathscr{I}_n\| \le K$, where K is defined in (A.6). We are done since $\sup_n \| \sqrt{\mathscr{I}_n} \| \le \sqrt{K}$, $\|\mathscr{I}\| \le K$, $\| \sqrt{\mathscr{I}} \| \le \sqrt{K}$. □

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