MTH/BIO 415 - Fall 2013

Homework Assignment #6: Nonlinear Differential Equations

Numerical Solutions: `ODE45`

Due: Wednesday, November 13

1. **SIRS Epidemic Models:** In class we discussed classic ODE models for the spread of infectious diseases. These are based on dividing the total population N into three classes: S: Susceptibles, I: Infected and R: Recovereds.

   Simple assumptions lead to the following model.

$$\begin{aligned}\frac{S}{dt} &= -\beta SI \\ \frac{I}{dt} &= -\beta SI - rI \\ \frac{R}{dt} &= rI\end{aligned}$$

   a) Explain, and using words and a diagram, the specific assumptions of this particular (SIR) model. What do the parameters β and r represent?

   b) The model assumes that everyone in the population N is in one of the three classes, mathematically: $N = S + I + R$. Rescale the variables in the model with the population size, N and derive new, scaled equations. What are the allowable limits of the new variables, $s = S/N$, $r = R/N$, $i = I/N$?

   c) Analyze the model using the $i - s$ phase plane. Find the null-clines and fix-points and the plot the direction fields. Sketch a few solutions.

   c) (Taken, verbatim, from `SIR MODELS: Module author: Florence Debarre Theoretical Biology Institute of Integrative Biology ETH Zurich`)

      A key parameter in epidemiology is the basic reproductive ratio, $R0$. It is defined as the average number of secondary cases transmitted by a single infected individual that is placed into a fully susceptible population. In other words, $R0$ tells us about the initial rate of spread of the disease. Hence, if $R0 > 1$, there will be an epidemic, and if $R0 < 1$, the introduced infecteds will recover (or die) without being able to replace themselves by new infections.

      To Do: Derive $R0$ for the simple SIR model. Use your phase plane analysis from part (b) to show, graphically, why this parameter predicts whether or not an epidemic can occur.

d) Write a *matlab* function for the basic SIR model. Use the package `ODE45` to find solutions for different paramter values and initial conditions. Plot $I(t)$ and $S(t)$ and also the solutions in the $S - I$ phase plane.

   In particular, use ODE45 to check the validity of the formula you derived for the onset of epidemics ($R0 > 1$). Fix N and chose combinations of β and r such that $R0 > 1$ and $R0 < 1$. Tabulate your results.

e) How would you implement VACCINATION in your model? Chose different proportions of Vaccinated individuals and run the model. Check whether the disease spreads. Attempt to determine the THRESHOLD vaccination percentage where for vaccination coverages lower than this number, the disease spreads.

f) What about TREATMENT? Model the treatment of infected individuals (assuming again life-long immunity). Implement a new Infected-Treated (IT) catagory. Let infected individuals initiate treatment at a certain rate, and let treated individuals recover at a faster rate compared to untreated infecteds. Model the effect of more and less effective drugs. What happens if treated individuals are also less infectious than untreated infecteds?

   Do this carefully: (1) Write down the model with the new IT class. (2) Write a `matlab` function to solve the new model equations. (3) Carefully compare what happens to the total number of infecteds over the course of the disease when (a) treatment has no effect (transmission and recovery rates for IT class are identical to the I class) (b) the recovery rate of IT is 10% faster than for I with the same transmission rate and (c) the transmission rate for IT is 10% lower than for I with the same recovery rate.

   Do this sort of thing for different diseases (different values of β and r)