NAME: **Math 330 - Exam 3 - May 3, 2018**

Please place your name on the top of the paper. Show your work in the space provided for each question.

Partial credit can only be given for work shown (!). GOOD LUCK!

1. (10 points) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ h & 6 \end{pmatrix}$$

   where h is a real number.

   (a) Find all values of h such that the matrix NOT *invertible* (singular)?

   (b) Find all values of h such that the matrix NOT *diagonalizable* ?

2. (15 points) If the statement is true, write **TRUE** with a brief justification. If it is false, then rewrite as a true stement.

   (a) A 10×10 matrix $\mathbf{A}$ has the property that $\mathbf{A}^t = \mathbf{A}$. This matrix is definitely diagonalizeable.

   (b) If $\mathbf{A}$ is an $n \times n$ diagonal matrix with nonzero diagonal elements, $a_{11}, a_{22}, \ldots, a_{nn}$, then $\mathbf{A}^{-1}$ is a diagonal matrix with entries $1/a_{11}, 1/a_{22}, \ldots 1/a_{nn}$.

   (c) The homogeneous system of equations, $\mathbf{AX} = 0$, $\mathbf{A}(n \times n)$ has a nontrivial solution when $\mathbf{A}$ is singular. (i.e. $\mathbf{A}^{-1}$ does not exist).

   (d) A 5×5 matrix $\mathbf{A}$ is found to have eigenvalues: $\lambda_1 = 1; \lambda_2 = 1; \lambda_3 = 1; \lambda_4 = -1; \lambda_5 = 3$.

      i. $\mathbf{A}$ is definitely non-singular.

      ii. $\mathbf{A}$ is definitely not diagonalizeable.

3. (20 points) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 2 & 0 \\ 0 & -2 & 6 \end{pmatrix}$$

(a) Find the eigenvalues and linearly independent eigenvectors of $\mathbf{A}$.

(b) Find the transformation matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$.

(c) Show how $\mathbf{P}$ could be used to calculate the matrix $\mathbf{A}^{32}$ efficiently. (No need to calculate - just show how!)

(d) Write down the fundamental solution matrix, $\mathbf{\Omega}(t)$ for the system:

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 2 & 0 \\ 0 & -2 & 6 \end{pmatrix} \mathbf{X}$$

4. (20 points) Consider the following homogeneous set of coupled, ordinary differential equations:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \quad A = \begin{pmatrix} -3 & -1 \\ 4 & -3 \end{pmatrix}$$

(a) Find the general solution, $\mathbf{X}(t)$.

(b) Write the general solution in terms of the fundamental matrix. $\boldsymbol{\Omega}(t)$.

5. (15 points) Consider the following homogeneous set of coupled, ordinary differential equations:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \quad \mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & 6 \end{pmatrix}$$

(a) Find the general solution, $\mathbf{X}(t)$.

(b) Write the general solution in terms of the fundamental matrix. $\boldsymbol{\Omega}(t)$.

6. (20 points) Consider the following non-homogeneous set of coupled, ordinary differential equations:

$$\mathbf{X}' = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$$

(a) Write the general solution to the homogeneous problem in terms of the fundamental solution matrix, $\Omega(t)$.

(b) Is the forcing function *resonant* with the homogeneous solution? Explain why or why not.

(c) Write the general solution to the non-homogeneous problem - Use EITHER diagonalization (Let $\mathbf{X} = \mathbf{PY}$) or Variation of Parameters.

(d) EXTRA CREDIT: Check your work by using the other method.

Extra extra-Credit: If there are $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{AB} = \mathbf{BA}$, PROVE that $(\mathbf{AB})(\mathbf{AB})(\mathbf{AB}) = \mathbf{A^3B^3}$.