Explicit Bounds for Primes in Arithmetic Progressions

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The Functions

$$\pi(x; q, a) = \sum_{p \le x} [\![p \equiv a \bmod q]\!]$$

$$\theta(x; q, a) = \sum_{p \le x} [\![p \equiv a \bmod q]\!] \log p$$

$$\psi(x; q, a) = \sum_{p^n \le x} [\![p^n \equiv a \bmod q]\!] \log p$$

All bounds flow from bounds on ψ.
Asymptotics known for a century.

Chebyshev-type Bounds

McCurley: $\theta(x; 3, a) < 0.51x$

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Many: $\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \epsilon_q \frac{x}{\phi(q)}, \quad x > e^L$

Each article has a page giving various combinations of q, ϵ, L

Many = McCurley, Ramaré-Rumely, Bennett, Dusart, Kadiri-Lumely, others

de la Vallée Poussin-type Bounds

Explicit bounds that are asymptotically correct

- $q = 1, 2$: Rosser-Schoenfeld, many others

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- $q = 3$: Dusart (2002) For $x \geq 151$,

$$\pi(x; 3, a) > \frac{1}{2} \frac{x}{\log x}.$$

de la Vallée Poussin-type Bounds

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$$\pi(x; 3, a) > \frac{1}{2}\frac{x}{\log x}.$$

- $q \geq 3$: Us (2018)

Our result

Theorem

Let $q \geq 3$ be an integer, and let a be coprime to q. Then

$$\left|\psi(x; q, a) - \frac{x}{\phi(q)}\right| < \frac{1}{160} \frac{x}{\log x} \quad \text{for all } x \geq x_0(q),$$

where

$$x_0(q) = \begin{cases} 8 \cdot 10^9 & \text{if } q \leq 10^5, \\ \exp(0.03\sqrt{q} \log^3 q), & \text{if } q > 10^5. \end{cases}$$

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Let $q \geq 3$ be an integer, and let a be coprime to q. Then

$$\left|\pi(x;q,a) - \frac{Li(x)}{\phi(q)}\right| < \frac{1}{160}\,\frac{x}{\log^2 x} \quad \textit{for all } x \geq x_0(q),$$

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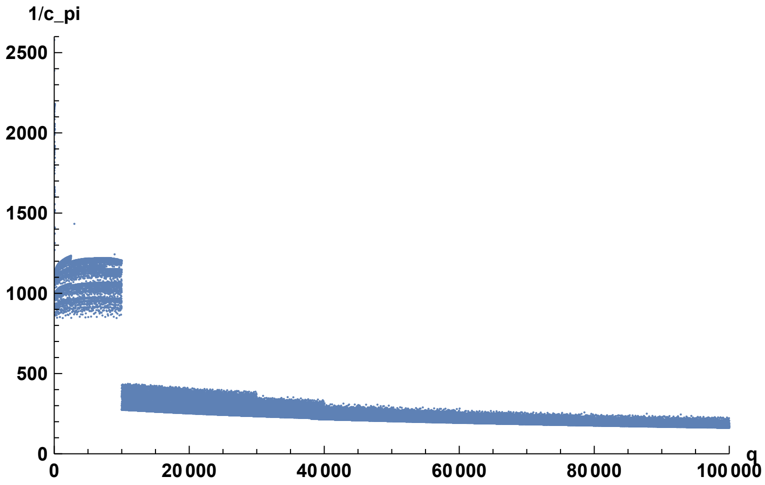
The constant as a function of q



The website

http://www.nt.math.ubc.ca/BeMaObRe/

Theorem

Let a be coprime to 10*, and* $x \geq 3\,375\,517\,771$*. Then*

$$\left|\pi(x; 10, a) - \frac{Li(x)}{\phi(q)}\right| < \frac{1}{2579}\,\frac{x}{\log^2 x}.$$

Let a be coprime to 3461*, and* $x \geq 9\,367\,751$*. Then*

$$\left|\pi(x; 3461, a) - \frac{Li(x)}{\phi(q)}\right| < \frac{1}{1204}\,\frac{x}{\log^2 x}.$$

Small q: Verifying GRH

GRH is equivalent to a certain quality error term, so results like this are billed as "verifying GRH".

Theorem

Let $q \leq 10^5$ and $(a, q) = 1$, and $x \leq 10^{11}$ (or larger if q is small). We have

$$\max_{1 \leq y \leq x} \left| \pi(y; q, a) - \frac{Li(y)}{\varphi(q)} \right| \leq 2.734 \frac{\sqrt{x}}{\log x}.$$

This is the computational limit giving $1/160$.

Small q: McCurley's Four-Terms Lemma

Let $x \geq x_2 > 2$, $H \geq 1$, $m \geq 2$ and $0 < \delta < \frac{x-2}{mx}$.
Suppose that every L-function mod q satisfies GRH(1). Then

$$\frac{\phi(q)}{x}\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{m\delta}{2} + U_{q,m,H}(x) + V_{q,m,H}(x) + W_q(x),$$

$$W_q(x) := \frac{\phi(q)}{x}\left(\left(\tfrac{1}{2} + \sum_{p|q}\tfrac{1}{p-1}\right)\log x + 4\log q + 13.4\right)$$

$$V_{q,m,H}(x) := \left(1 + \frac{m\delta}{2}\right)\sum_{\chi}\sum_{\substack{\rho \\ |\gamma|\leq H}}\frac{x^{\beta-1}}{|\rho|}$$

$$U_{q,m,H}(x) := \frac{A_m(\delta)}{\delta^m}\sum_{\chi}\sum_{\substack{\rho \\ |\gamma|>H}}\frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}$$

Local Effects are negligible

$$W_q(x) := \frac{\phi(q)}{x}\left(\left(\frac{1}{2} + \sum_{p|q} \frac{1}{p-1}\right)\log x + 4\log q + 13.4\right)$$

Tiny.

Low Height Tools

$$V_{q,m,H}(x) := \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\substack{\rho \\ |\gamma| \le H}} \frac{x^{\beta-1}}{|\rho|}$$

- sum over all characters mod q
- $\rho = \beta + i\gamma, L(\rho, \chi) = 0$

Low Height Tools

$$V_{q,m,H}(x) := \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\substack{\rho \\ |\gamma| \le H}} \frac{x^{\beta-1}}{|\rho|} < \left(1 + \frac{m\delta}{2}\right) \frac{C_q}{\sqrt{x}}$$

- sum over all characters mod q
- $\rho = \beta + i\gamma, L(\rho, \chi) = 0$
- Platt: verified GRH for $H = 10^8/q$ with $q \le 10^5$
- Rubinstein: `lcalc`, fast reliable computation of zeros
- Rosser: $N(T) < \frac{T}{\pi} \log \frac{T}{2\pi e} + 0.34 \log T + 6$
- Trudgian: $N(T, \chi) < \frac{T}{\pi} \log \frac{q^* T}{2\pi e} + 0.4 \log(q^* T) + 5.4$

Large Height Bound

$$U_{q,m,H}(x) := \frac{A_m(\delta)}{\delta^m} \sum_{\chi} \sum_{|\gamma|>H} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}$$

$$< \frac{A_m(\delta)}{\delta^m} \sum_{\chi} \sum_{|\gamma|>H} \frac{1}{|\rho|^{m+1}}$$

$$\approx \left(\frac{2}{\delta}\right)^m \cdot C_q$$

Large Height Bound

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$$\approx \left(\frac{2}{\delta}\right)^m \cdot C_q$$

Where are we?

$$\frac{\phi(q)}{x}\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{m\delta}{2} + \left(\frac{2}{\delta}\right)^m \cdot C_q + (1 + \frac{m\delta}{2})\frac{C_q}{\sqrt{x}} + \text{tiny}$$

Zero-free region

Suppose that $L(\beta + i\gamma, \chi) = 0$, and $\beta > 0$.
Then

$$\beta \leq 1 - \frac{1}{R \log(q^* |\gamma|)}$$

- McCurley: $R = 9.645908801$
- Kadiri: If $3 \leq q \leq 4 \cdot 10^5$, then $R = 5.6$
- Platt: GRH

Revised McCurley

What happens with $q \geq 10^5$, without GRH?

McCurley's Four-terms Lemma needs to be adjusted! The main concern is the potential existence of an exceptional zero. Absorbing such concerns into $W_q(x)$ gives:

$$\tilde{W}_q(x) < \frac{\varphi(q)}{x} \left(\frac{\log q \cdot \log x}{\log 2} + 0.2516 q \log q \right).$$

$L(1, \chi)$

Theorem

Let χ be a primitive quadratic character with modulus $q > 6677$. Then

$$L(1, \chi) \geq \frac{12}{\sqrt{q}}.$$

If $q \geq 4 \cdot 10^5$, then

$$L(1, \chi) \geq \frac{1}{\sqrt{q}} \min \left\{ 46\pi, \max\{12, \log \frac{\sqrt{q+4} + \sqrt{q}}{2}\} \right\}.$$

How exceptional is an Exceptional Zero?

Theorem

Let $q \geq 3$*, and* χ *a quadratic character modulo* q*. If* $\beta > 0$ *and* $L(\beta, \chi) = 0$*, then*

$$\beta \leq 1 - \frac{40}{\sqrt{q}\log^2 q}.$$

The $x_2(q)$

We can get any positive number we want in place of "$\frac{1}{160}$" by making x_2 (where the bound becomes official) large enough.
For any positive number, however, x_2 is intractably large.

Small x

Theorem

Suppose that $3 \le q \le 10^5$, $\gcd(a, q) = 1$, *and* $x \ge 1000$. *Then*

$$\left|\psi(x; q, a) - \frac{x}{\varphi(q)}\right| < 0.19 \frac{x}{\log x}$$
$$\left|\theta(x; q, a) - \frac{x}{\varphi(q)}\right| < 0.40 \frac{x}{\log x}$$
$$\left|\pi(x; q, a) - \frac{Li(x)}{\varphi(q)}\right| < 0.53 \frac{x}{\log^2 x}$$

Small q, small x

Theorem

Let $3 \le q \le 1200$ be an integer, and let a be an integer coprime to q. For all $x \ge 50q^2$

$$\frac{1}{\phi(q)} \frac{x}{\log x} < \pi(x; q, a)$$

and

$$\pi(x; q, a) < \frac{1}{\phi(q)} \frac{x}{\log x} \left(1 + \frac{5}{2 \log x}\right).$$

The n-th prime in an AP

Theorem

Let $3 \le q \le 1200$ be an integer, and let a be an integer coprime to q. If $p_n(q,a) \ge 22q^2$, then $p_n(q,a)$ is between

$$N \log N$$

and

$$N \log N + \frac{4}{3} N \log \log N,$$

where $N = n\varphi(q)$.

Meta Issues

A 100 page technical argument, with large chunks pushed through by computation. What could go wrong?

- How to avoid small errors?
  1. Special Cases
  2. Redundant Coding
  3. Vigorous self-refereeing
  4. Careful Writing
- How to inspire confidence that small errors have been avoided?
  1. Giving detailed code
  2. Giving detailed data
- Other
  1. Open source
  2. Easy to extend? Not so much...yet.

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