Reciprocals of Binary Power Series

Joshua N. Cooper
Institute of Theoretical Computer Science
Zürich, Switzerland

Dennis Eichhorn
California State University, East Bay

Kevin O'Bryant
City University of New York, College of Staten Island

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If $R(k) > 0$, then

$$k = 2^n + 2^m - 1,$$

and if $n \neq m$

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so $R(k) = 2$.

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If $R(k)>0$, then

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If $n=m$, then

$$k=(2^n)+(2^m-1)=(0)+(2^{n+1}-1),$$

and so $R(k)=2$.

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Nonnegative integer sets A and B are reciprocals if their generating functions are reciprocals in $\mathbb{F}_2[[q]]$.

$$A = \{0,1\}, \quad B = \{0,1,2,3,\ldots\}$$

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Remark: $\mathcal{F} \in \mathbb{F}_2[[q]]$ is invertible if and only if...

If $\max A = d$, then

$$b_n = b_{n-1}a_1 + b_{n-2}a_2 + \cdots + b_{n-d}a_d.$$

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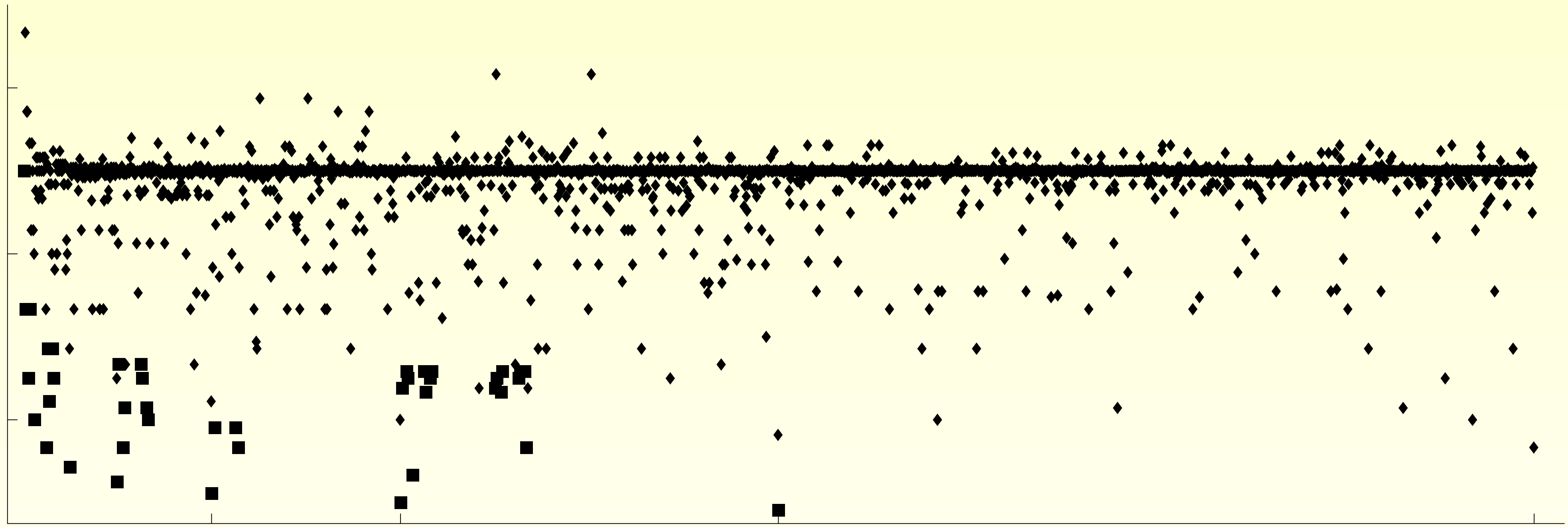
- (b) is periodic.
- (b) may have more 0 than 1 (when d is small)
- If q generates the multiplicative group of $\mathbb{F}_2[q]/(\mathcal{A})$, then every binary word of length d appears in (b) except $0000\cdots000$. This is called a reduced de Bruijn cycle.
- Period length $= 2^d - 1$, with 2^{d-1} ones. Density slightly larger than $1/2$.



The points $(n, \delta(\bar{\mathcal{P}}_n))$, where the coeffs of $\mathcal{P}_n$ are the binary expansion of n.

- What are the possible densities of reciprocals of finite sets?
- Is the bias toward $< 1/2$ a law of small numbers?

If $\mathcal{P}(q)$ is a polynomial, then there is another polynomial $\mathcal{P}^*$ and a positive integer D such that $\mathcal{P}\mathcal{P}^* = 1 + q^D$. We call the minimal such D the order of $\mathcal{P}$.

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Proposition: The reciprocal of an eventually periodic set is one too.

$$\Theta(c_1, c_2) := \left\{ c_1 n + c_2 \frac{n(n-1)}{2} : n \in \mathbb{Z} \right\}$$

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$$\Theta(0,1) = \left\{ \binom{n}{2} : n \ge 1 \right\}$$

$$\Theta(1,2) = \{n^2 : n \ge 0\}$$

$$\Theta(1,3) = \{\text{pentagonals}\}$$

c_2 \ c_1	1	2	3	4	5	6
2	2090					
3	5004					
4	5088					
5	5057	5019				
6	2114					
7	5020	5023	5000			
8	5002		5045			
9	5085	4942		4994		
10	3854		4062			
11	4994	4959	5073	4982	5039	
12	5044				5073	
13	4985	5002	4973	5071	4963	5090
14	4391		4445		4109	

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The reciprocal of the set $\Theta(c_1, c_2)$, where $0 \le 2c_1 \le c_2$ and $\gcd(c_1, c_2) = 1$, has density 0 if $c_2 \equiv 2 \pmod 4$, and otherwise has density 1/2.

More precisely, if $c_2 \equiv 2 \pmod 4$, then

$$\lim_{n\to\infty} \frac{\left|\overline{\Theta(c_1, c_2)} \cap [0, n]\right|}{n/\log n} = C,$$

for some positive constant C depending only on c_2. If $c_2 \not\equiv 2 \pmod 4$, then

$$\limsup_{n\to\infty} \left|\frac{\left|\overline{\Theta(c_1, c_2)} \cap [0, n]\right| - n/2}{\sqrt{n \log\log(n)/2}}\right| = 1.$$

How many numbers less than N can be written in the form

$$x_0^2 + 2x_1^2 + 4x_2^2 + 8x_3^2 + 16x_4^2 + \cdots ,$$

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Current bests:

$\# \geq \left(\dfrac{\pi^2\sqrt{3}}{2} - o(1) \right) \dfrac{\sqrt{N}}{\log N}$	(D. Eichhorn)
$\lim_{N\to\infty} \frac{N-\#}{\sqrt{N}} = \infty$	(Serre)

Let $f_1, f_2, \dots$ be independent binary random variables, with

$$\mathbb{P}[f_n = 0]\mathbb{P}[f_n = 1]$$

bounded away from 0.

Define $\bar{f}_1, \bar{f}_2, \dots$ by

$$(1 + f_1 q + f_2 q^2 + f_3 q^3 + \cdots)(1 + \bar{f}_1 q + \bar{f}_2 q^2 + \bar{f}_3 q^3 + \dots) = 1.$$

Then the number of $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_N$ that are 1 is $\sim N/2$ with probability 1.

$$\bar{f}_n = \sum_{\vec{x}} f_{x_1} f_{x_2} \cdots f_{x_\ell}$$

where the summation extends over all tuples $\vec{x} = (x_1, \ldots, x_\ell)$ with $n = \sum_{i=1}^{\ell} x_i$ and each $x_i > 0$ (ℓ is allowed to vary).

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$$\bar{f}_n = f_n + f_{n-2i}\bar{f}_i + f_{n-4i}\bar{f}_2 + \ldots f_{n/2}\bar{f}_{n/4} + mess$$

and *mess* depends only on $f_1, f_2, \ldots, f_{n/2-1}$.

Thus,

$$H[f_n|f_1,\ldots,f_{n/2-1}] \geq H[\sum_{i\in A} f_i|A]$$

where $A = \{n-2i\colon 0 \leq i < n/4, \bar{f}_i = 1\}$. Since

- $|A| \to \infty$ (requires easy proof),
- this uncertainty goes to 1/2 (requires proof),
- and so $\mathbb{P}[f_n = 0] \to 1/2$ (obvious),
- and consequently $\#\{n \leq N\colon \bar{f}_n = 0\} \sim N/2$ (obscure Borel-Cantelli Lemma)

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Plans for future development:

- Take some interesting set of integers, call it A. Find $\bar{A}$.
- Probabilistic argument is not most general possible.
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