8.41 $Y^2/\sigma^2 \sim \chi^2(1)$. So $Y^2/u \le \sigma^2 \le Y^2/l$.

a) $[0.2 \cdot Y^2, 1018 \cdot Y^2]$ using R and the qchisq function

b) $254.3 \cdot Y^2$

c) $0.26 \cdot Y^2$

8.42 a) $[0.44 \cdot Y, 31.9 \cdot Y]$

b) $15.9 \cdot Y$

c) $0.51 \cdot Y$

8.62 a) $\bar{X} - \bar{Y} = 24.8 - 21.3 = 3.5$

$\sigma_X^2/n_X = 1.4826$

$\sigma_Y^2/n_Y = 1.6002$

$\sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{n}} = 1.7558$

99% $z_l = F^{-1}(.005) = -2.5758$ $z_u = 2.5758$

So: $(-1.02, 8.02)$

b) Normal males molt in anything from 1 day less to 8 days more than split males. At a 99% confidence level we cannot see any systematic differences.

8.70 At 95%, the multiplier is 1.96. So to get a width of ± 0.05, we need $1.96 \cdot \sqrt{pq/n} \le .05$

$\sqrt{pq/n} \le .025$

$\sqrt{pq} \le .025\sqrt{n}$

$pq/.000625 \le n$

$p \approx .9$ $n \geq 144$

p anything then $pq \le \frac{1}{4}$.

$n \geq 400$.

8.86 Using t.test in R:

a) mean 0.896
df 13
CI 0.665 — 1.127

b) mean 1.147
df 10
CI 0.691 — 1.603

- water has more observations
- water has smaller variance

8.87 using t.test — uses Welch's dof, which will give different numbers from doing it by hand.

CI −0.655 — 0.153

CI contains 0, so "no difference" is plausible.

8.102

ages	39	54	61	72	59
squares	1521	2916	3721	5184	3481

sum of squares 16823

CI $= \sqrt{SS/\chi^2_{n,\alpha}} = (33, 285)$

Exponential Families

Here is one way to create a parametrized set of distributions: pick a density in the form:

$$f(x) = \exp\left[\eta(\vartheta)^T \cdot T(x) + A(\vartheta) + B(x)\right] =$$
$$= \exp\left[\eta(\vartheta)^T \cdot T(x)\right] \cdot g(\vartheta) \cdot h(x)$$

Example

$N(\mu, \sigma^2)$ has PDF $\dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right] =$

$$= \exp\left[-\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma^2 - \frac{1}{2}\frac{x^2}{\sigma^2} + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}\right] =$$

$$= \exp\left[\underbrace{\begin{pmatrix} -\frac{1}{2\sigma^2} \\ \frac{\mu}{\sigma^2} \end{pmatrix}^T}_{\eta} \underbrace{\begin{pmatrix} x^2 \\ x \end{pmatrix}}_{T} \underbrace{- \frac{1}{2}\left(\log\sigma^2 + \frac{\mu^2}{2\sigma^2}\right) - \frac{1}{2}\log 2\pi}_{A}\right]$$

Example

The Poisson distribution with rate λ has probability density

$$f(k) = \frac{\lambda^k e^{-\lambda}}{k!} = \exp\left[k\log\lambda - \lambda - \sum_{i=1}^{k}\log i\right] =$$

$$= \exp\left[\underbrace{k}_{T}\ \underbrace{\log\lambda}_{\eta}\right] \cdot \underbrace{e^{-\lambda}}_{g} \cdot \underbrace{\frac{1}{k!}}_{h}$$

Also exponential families

Distribution	$T(x)$	$\eta(\vartheta)$
Bernoulli	x	$\log \frac{p}{1-p}$
Binomial (n)	x	$\log \frac{p}{1-p}$
Exponential	x	$-\lambda$
χ^2	$\log x$	$\frac{d}{2}-1$

$g(\vartheta)$ or also $A(\vartheta)$ captures the constant needed to convert the rest of the expression into a prob. density.

It also produces the moment generating function of $T(x)$:

$$M_T(u) = \exp\left[A(\eta+u) - A(\eta)\right]$$

The cumulant generating function is

$$K_T(u) = \log M_T(u) = A(\eta+u) - A(\eta)$$
$$= \mu t + \sigma^2 \frac{t^2}{2} + \dots$$

CGFs are additive!

If $X_1, \dots, X_n \sim \exp[\eta^T T]\, g(\vartheta)\, h(x)$

Then the joint density is also an exponential family:

$$\exp\left[\eta^T \left(\Sigma T(x_i)\right)\right] g(\vartheta)^n \prod h(x_i)$$

In an exponential family, the vector $\eta(\vartheta)$ is called the natural parameter and the vector $T(x)$ is the sufficient statistic.

This is a first example of sufficiency:

Def A statistic $T = T(X_1, \dots, X_n)$ is sufficient for ϑ if $\mathbb{P}(X_1, \dots, X_n \mid T, \vartheta) = \mathbb{P}(X_1, \dots, X_n \mid T)$, (discrete)
$\left(f(x_1, \dots, x_n \mid T, \vartheta) = f(x_1, \dots, x_n \mid T). \right)$ (continuous)

This probability/density is going to be important enough to warrant a name:

Def The likelihood of ϑ given $X_1, \dots, X_n$ is

$$\mathcal{L}(\vartheta \mid x_1, \dots, x_n) = \begin{cases} \mathbb{P}(x_1, \dots, x_n \mid \vartheta) & \text{(disc.)} \\ f(x_1, \dots, x_n \mid \vartheta) & \text{(cont.)} \end{cases}$$

Theorem $T(x)$ is sufficient for ϑ if

$$\mathcal{L}(\vartheta \mid x) = g(T, \vartheta) \cdot h(x).$$

Here, ϑ and x only interact directly through $T(x)$. Any non-T x shows up isolated by multiplication from ϑ.

Example $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$ Then their joint density is

$$L(\lambda | x_1, \dots, x_n) = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod x_i!} = \underbrace{\lambda^{\sum x_i} e^{-n\lambda}}_{g} \cdot \underbrace{\frac{1}{\prod x_i!}}_{h}$$

Here the sufficient statistic is $\sum x_i$.

Theorem For an exponential family, $T(x)$ is a sufficient statistic.

Proof

$$g(T, \vartheta) = \exp\left[\eta(\vartheta)^T \cdot T + A(\vartheta)\right]$$

$$h(x) = \exp[B(x)]$$

is a factorization of the exp. fam. density function.

$\blacksquare$