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Today we need a few core results about the t- and χ^2-distributions.

χ^2 PDF $f_{\chi^2(k)}(x) = C \cdot x^{k/2-1} \cdot \exp[-x/2]$

Mean k. Variance $2k$

The main place that $\chi^2(k)$ appears is in the sums of squares of normal variables:

<u>Theorem</u> $Y_1, \ldots, Y_n \sim \mathcal{N}(\mu, \sigma)$ iid. Then $Z_i = \frac{Y_i - \mu}{\sigma} \sim \mathcal{N}(0,1)$ and

$$\sum Z_i^2 \sim \chi^2(n).$$

The proof uses moment-generating functions:

<u>Dfn</u> The k^{th} <u>moment of a random variable</u> Y is $\mu_k' = \mathbb{E}[Y^k]$.

The <u>k^{th} central moment</u> is $\mu_k = \mathbb{E}[(Y-\mu_1')^k]$

The <u>moment-generating function</u> is $m(t) = \mathbb{E}[\exp[tY]]$

Series expanding we get $m(t) = \sum \exp[ty] p(y) =$

$= \sum \left(1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \ldots\right) p(y) = \sum p(y) + t \sum y p(y) + \frac{t^2}{2!} \sum y^2 p(y) + \ldots$

$= \sum \frac{t^n}{n!} \mu_n'$. So the k^{th} coefficient is the k^{th} moment!

Theorem If X, Y are different random variables, then if $m_X(t) = m_Y(t)\ \forall t$, then X, Y have the same distribution.

Proof omitted. Note: $m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$ for independent X, Y. bc exponent laws

Moment generating functions can be used to prove a range of useful theorems and calculations.

$N(\mu, \sigma)$ has $m(t) = \exp\left[\mu t + \sigma^2 \frac{t^2}{2}\right]$

Theorem (Central Limit Theorem)

Suppose $Y_1, \dots, Y_n$ iid with finite μ_1' and μ_2.

Write $U_n = \dfrac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$. Then $U_n \xrightarrow[n\to\infty]{} N(0,1)$.

Proof $U_n = \frac{1}{\sqrt{n}} \sum Z_i$ with $Z_i \sim N(0,1)$.

$$m_{\sum Z_i}(t) = m_{Z_1}(t) \cdot \dots \cdot m_{Z_n}(t) = [m_{Z_1}(t)]^n.$$

$$m_{U_n}(t) = m_{\frac{1}{\sqrt{n}}\sum Z_i}(t) = E \exp\left[t \cdot \frac{1}{\sqrt{n}} \cdot \sum Z_i\right] = m_{\sum Z_i}\left(\frac{t}{\sqrt{n}}\right) = \left[m_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

Taylor expand: $m_Z(t) = m_Z(0) + m_Z'(0)\, t + m_Z''(\xi) \frac{t^2}{2}$

Here: $m_Z(0) = 1$

$m_Z'(0) = EZ = 0$

So $$m_{U_n}(t) = \left[1 + \frac{m_Z''(\xi)}{2} \left(\frac{t}{\sqrt{n}}\right)^2\right]^n = \left[1 + \frac{m_Z''(\xi)\, t^2/2}{n}\right]^n$$

As $n \to \infty$, $\xi \to 0$; $m_Z''(\xi_n)\, t^2/2 \to m_Z''(0)\, t^2/2 = t^2/2$

since $m''(0) = \mu_2' = \mu_2 = \sigma^2 = 1$.

bc $\mu = 0$

Recall that if $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} \left(1 + \frac{b_n}{n}\right)^n = e^b$.

Hence $\lim_{n\to\infty} m_{U_n}(t) = \lim \left[1 + \frac{m_Z''(\xi)\, t^2/2}{n}\right]^n = \exp[t^2/2]$.

But this is the mgf for $\mathcal{N}(0,1)$. ∎

Theorem $Z \sim \mathcal{N}(0,1)$. Then $Z^2 \sim \chi(1)$.

Proof (Example 6.11)

$$m_{Z^2}(t) = \mathbb{E}\exp[tZ^2] = \int_{-\infty}^{\infty} e^{tz^2} f(z)\,dz = \int_{-\infty}^{\infty} e^{tz^2} \cdot \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left[-\frac{z^2}{2} \cdot (1-2t)\right] dz$$

This is proportional to the density function for

$\mathcal{N}\left(0, \sqrt{(1-2t)^{-1}}\right)$

So $m_{Z^2}(t) = \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \text{density function}\, dz = (1-2t)^{-1/2}$.

This is the mgf for a $\Gamma(\frac{1}{2}, 2)$ distributed variable, i.e a $\chi^2(1)$ distribution. ∎

This used, inter alia:

$$m_{\Gamma(\alpha,\beta)}(t) = (1-\beta t)^{-\alpha}.$$

Also note
- $\Gamma(1,\lambda) = \text{Exponential}(\lambda)$
- $\Gamma(\frac{\nu}{2}, 2) = \chi^2(\nu)$
- If $X_i \sim \Gamma(\alpha_i, \vartheta)$ then $\sum X_i \sim \Gamma(\sum \alpha_i, \vartheta)$.

Theorem $\sum Z_i^2 \sim \chi^2(n)$.

Proof See above. $\blacksquare$

Theorem $\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum(Y_i - \bar{Y})^2 \sim \chi^2(n-1)$.

Proof sketch The $Y_i - \bar{Y}$ are rewritten from n summands into $n-1$ summands of $N(0,\sigma)$ variables.
Dividing by σ^2 rescales to a sum of squares of $N(0,1)$ variables. $\blacksquare$

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$t(n)$ The $t(n)$ distribution has

$$f(y) \propto \left(1 + \frac{y^2}{n}\right)^{-(n+1)/2}$$

mean 0 for $n > 1$

variance $\begin{cases} n/(n-2) & \text{for } n > 2 \\ \infty & \text{for } n \in (1,2] \end{cases}$

It was popularized by A. Student (William Gosset) and Ronald Fisher; first described in 1876.

Theorem $Z \sim N(0,1)$, $V \sim \chi^2(n)$ independent.

Then

$$T = \frac{Z}{\sqrt{V/n}} \sim t(n).$$

Proof omitted.

Theorem $Y_1, \dots, Y_n \sim N(\mu, \sigma^2)$

Then $\dfrac{\bar{Y}-\mu}{S/\sqrt{n}} \sim t(n-1)$.

Proof $\dfrac{\bar{Y}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$. Also $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

Now, $\dfrac{\bar{Y}-\mu}{\sqrt{\sigma^2/n}} \Bigg/ \sqrt{\dfrac{(n-1)S^2}{\sigma^2} \Big/ n-1} \sim t(n-1)$.

But here: $\dfrac{\bar{Y}-\mu}{\sqrt{\sigma^2/n}} \Bigg/ \sqrt{\dfrac{\cancel{(n-1)}S^2}{\sigma^2} \Big/ \cancel{n-1}} = \dfrac{(\bar{Y}-\mu)}{\cancel{\sqrt{\sigma^2}}/\sqrt{n}} \cdot \sqrt{\dfrac{\cancel{\sigma^2}}{S^2}} =$

$$= \frac{\bar{Y}-\mu}{S/\sqrt{n}}. \qquad \blacksquare$$

Now we have a pivot for μ !!

For normally distributed input, anything we do with this pivot is accurate down to very low n. For non-normal, it has turned out to be remarkably resilient.

Historical problem: different lookup tables for each n.
Nowadays: Use computers! Even Excel can calculate with the t distribution !!

For each sample mean used, the DoF drops by one: so when using the $\overline{Y}_1 - \overline{Y}_2$ type estimators, the T-score follows a $t(n_1+n_2-2)$ distribution.

Exercise 8.93

CIs for variance

Large sample S^2 is a sum (of squared deviations) so will asymptotically follow a normal distr. Hence the Wald approach works.

Small sample $Y_1, \dots, Y_n \sim N(\mu, \sigma^2)$ both μ and σ^2 unknown.

Then we know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

This is a pivot!

Exercise Given χ^2_L and χ^2_u, write out the CI.

χ^2 is not symmetric (cannot be, since bounded below by 0, but unbounded above...)

So different choices in χ^2_u, χ^2_L

- equal tail areas
- shortest interval (difficult!)

Note: T intervals can tolerate non-normal data
χ^2 intervals are sensitive to non-normality.