p-Values

For a test with rejection region determined as $\{T>k\}$ for some test statistic T, we can pick any k — achieving different levels α.

For a particular value t of T, we get a split of the interval $[0,1]$ into $\{\alpha : t \text{ rejects the null}\} \cup \{\alpha : t \text{ accepts}\}$.

<u>Definition</u> Given a test statistic T and a realization t of T, the <u>p-value</u> is the smallest level α for which the test rejects the null for $T=t$.
This is also called the <u>attained significance level</u>.

For most if not all tests* we meet in this course, the test is constructed by creating a test statistic whose distribution under the null hypothesis is well known.

*except the upcoming likelihood ratio tests

In these cases, extremal ranges are quite easy to compute — they correspond to evaluating the distribution function.

Indeed, if the rejection region is $\{T \geq k\}$ and we have observed $T=t$, then $k=t$ produces the largest rejection region, and hence the smallest α.
This attained significance α then is $\mathbb{P}(T \geq t) = 1 - F_T(t)$.

Similarly, for a rejection region $\{T \leq k\}$, we get $p = \mathbb{P}(T \leq t) = F_T(t)$.

Finally, for two-tailed tests, the setup would be a rejection region of $\{|T| \geq k\}$, which has $|t|$ as the boundary value, yielding

$$p = \mathbb{P}(|T| \geq |t|) = \mathbb{P}(T \leq -|t|) \cup \mathbb{P}(T \geq |t|) =$$

$$= F_T(-|t|) + (1 - F_T(|t|)) \underset{\substack{\text{for symmetric} \\ \text{around } 0}}{=} 2F_T(-|t|).$$