Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Spring 2019

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

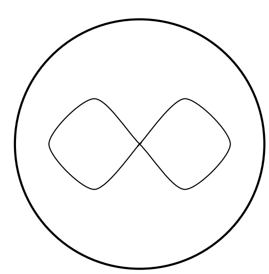
1. Let A be a non-empty closed subset of the metric space X and define $f(x) = \inf\{d(x,a) \mid a \in A\}$. Show that $f(x) = 0$ if and only if $x \in A$.
2. Show that every closed subset of a complete metric space is complete.
3. Let $A \subset \mathbb{R}^2$ be countable. Show that $\mathbb{R}^2 - A$ is connected.
4. Prove that if a compact connected Hausdorff space is countable, then it has exactly one point.

Part II

5. Explicitly construct all 2-fold covers of $S^1 \vee S^1$.
6. Let X be the two-dimensional torus with one point removed and let Y be the two-dimensional sphere with three points removed.
   (a) Do X and Y have the same homotopy groups?
   (b) Are X and Y homeomorphic?
7. Let X be the cell complex obtained by attaching two 2-cells to S^1 by the attaching maps $z \to z^3$ and $z \to z^2$.
   (a) Compute $\pi_1(X)$.
   (b) Is X homeomorphic to S^2?
8. (a) Show that any map $f : \mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotopic.
   (b) Find, with proof, a map $g : S^1 \times S^1 \to \mathbb{R}P^2$ which is not nullhomotopic.
9. Let X be the union of the unit sphere in the 3-space with the straight line segment from the north pole to the south pole. Compute $\pi_1(X)$.

Part III

10. Let X be the space formed by gluing the boundary of a disc to the interior of a disc along a figure-8 curve, as illustrated in the figure. Write down an explicit cell structure for X, and use it to compute $H_*(X)$.



11. Write down an explicit Δ-complex structure on the Klein bottle, and use it to compute the cup product structure on cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.
12. Show that $S^2 \vee S^4$ is not homotopy equivalent to a manifold.

13. Let $\Delta_n = \{(x_0, x_1, \ldots, x_n) \mid x_i \geq 0 \ \forall i = 0 \ldots, n, \text{ and} \sum_{i=0}^{n} x_i = 1\}$ be the standard n-simplex. Let $\Delta_n^{(k)}$ denote its k-skeleton. Compute the homology groups $H_*(\Delta_4^{(2)})$.

14. Let $X = S^2/\{p, q\}$ be the 2-sphere with the points p and q identified. Compute the following with $\mathbb{Z}$ co-efficients.

    (a) Compute the local homology groups $H_*(X, X - x)$ when $x = [p]$.

    (b) Compute the local homology groups $H_*(X, X - x)$ when $x \neq [p]$.

    (c) Compute the homology groups $H_*(X)$.