### **Topology Qualifying Exam**

#### Mathematics Program, CUNY Graduate Center

# Fall 2019

**Instructions:** Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

# Part I

- 1. Let A be a non-empty closed subset of the metric space X and define  $f(x) = \inf\{d(x, a) \mid a \in A\}$ . Show that f(x) = 0 if and only if  $x \in A$ .
- 2. Show that every closed subset of a complete metric space is complete.
- 3. Let  $A \subset \mathbb{R}^2$  be countable. Show that  $\mathbb{R}^2 A$  is connected.
- 4. Prove that if a compact connected Hausdorff space is countable, then it has exactly one point.

## Part II

- 5. Explicitly construct all 2-fold covers of  $S^1 \vee S^1$ .
- 6. Let X be the two-dimensional torus with one point removed and let Y be the two-dimensional sphere with three points removed.
  - (a) Do X and Y have the same homotopy groups?
  - (b) Are X and Y homeomorphic?
- 7. Let X be the cell complex obtained by attaching two 2-cells to  $S^1$  by the attaching maps  $z \to z^3$  and  $z \to z^2$ .
  - (a) Compute  $\pi_1(X)$ .
  - (b) Is X homeomorphic to  $S^2$ ?
- 8. (a) Show that any map  $f : \mathbb{R}P^2 \to S^1 \times S^1$  is nullhomotopic.
  - (b) Find, with proof, a map  $g: S^1 \times S^1 \to \mathbb{R}P^2$  which is not nullhomotopic.
- 9. Let X be the union of the unit sphere in the 3-space with the straight line segment from the north pole to the south pole. Compute  $\pi_1(X)$ .

### Part III

10. Let X be the space formed by gluing the boundary of a disc to the interior of a disc along a figure-8 curve, as illustrated in the figure. Write down an explicit cell structure for X, and use it to compute  $H_*(X)$ .



- 11. Write down an explicit  $\Delta$ -complex structure on the Klein bottle, and use it to compute the cup product structure on cohomology with  $\mathbb{Z}/2\mathbb{Z}$  coefficients.
- 12. Show that  $S^2 \vee S^4$  is not homotopy equivalent to a manifold.

- 13. Let  $T_1$  and  $T_2$  be solid tori  $D^2 \times S^1$ . Let  $X = T_1 \cup_f T_2$  be the 3-manifold obtained by the attaching map  $f: \partial D^2 \times S^1 \to \partial D^2 \times S^1$ , given by f(x, y) = (x, y); i.e., attach the solid tori by identifying their meridians. Use the Mayer-Vietoris sequence to compute the homology of X.
- 14. Let  $X = S^2/\{p,q\}$  be the 2-sphere with the points p and q identified. Compute the following with  $\mathbb{Z}$  co-efficients.
  - (a) Compute the local homology groups  $H_*(X, X x)$  when x = [p].
  - (b) Compute the local homology groups  $H_*(X, X x)$  when  $x \neq [p]$ .
  - (c) Compute the homology groups  $H_*(X)$ .