Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Fall 2018

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Prove that if $\{X_i \,|\, i \in I\}$ is a family of connected spaces such that $\cap_{i\in I} X_i \neq \emptyset$ then $\cup_{i\in I} X_i$ is connected.

2. Let A be a non-empty subset of the metric space X and define $f(x) = \inf\{d(x,a) \mid a \in A\}$. Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

3. Let X be a topological space. A family $\mathcal{F}$ of subsets of X is called *locally finite* if every point $x \in X$ has an open neighborhood U such that only finitely many members of $\mathcal{F}$ have nonempty intersection with U. Prove that if $\mathcal{F}$ is a locally finite family of closed sets then the union $\cup_{C\in\mathcal{F}}$ is closed.

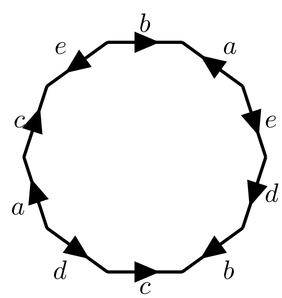
4. Let τ be the standard topology on the unit interval $I = [0,1]$ and let τ' be another topology on I.

   (a) Prove that if $\tau' \subsetneq \tau$ then I cannot be Hausdorff with the topology τ'.

   (b) Prove that if $\tau \subsetneq \tau'$ then I cannot be compact with the topology τ'.

Part II

5. State the classification of closed orientable surfaces. Compute the Euler characteristic of the surface obtained by identifying the sides of the polygon drawn below, and identify which surface it is.



6. Consider $\mathbb{R}^3$ with both the z-axis removed and the unit circle in the xy-plane removed. What is the fundamental group of this space.

7. (a) Show directly that every map from S^1 to S^2 is null-homotopic.

   (b) Show that every map from S^2 to S^1 is null-homotopic.

8. Explicitly enumerate all 3-fold covers of the Klein bottle.

Part III

9. Let M be a closed, connected, orientable 4–manifold with $\pi_1(M) \cong \mathbb{Z}_3 * \mathbb{Z}_3$ and $\chi(M) = 5$.

   (a) Compute $H_i(M, \mathbb{Z})$ for all i.

(b) Let X be a CW-complex with no 3–cells. Show that M is not homotopy equivalent to X.

10. Let $X = S^1 \vee S^1 \vee S^2$.

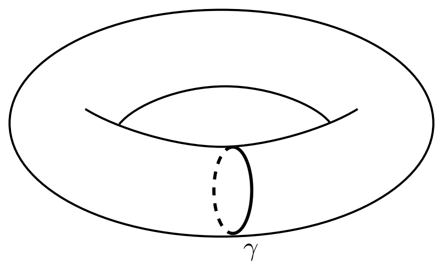
    (a) Show that X and the 2-torus T^2 have CW-structures with four cells: one 0-cell, two 1-cells and one 2-cell.

    (b) Use cellular homology to show that T^2 and X have isomorphic homology groups.

    (c) Use cup products to show that T^2 and X are not homeomorphic.

11. Calculate the cohomology ring of $\mathbb{CP}^2$. Then calculate the cohomology ring of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

12. The space X is obtained by attaching a Möbius strip along its boundary to one of the meridians of a two-torus (marked γ in the picture below). Use the Mayer-Vietoris theorem to compute the homology of X.



13. Let S and S' be closed orientable surfaces. A map $f\colon S \to S'$ induces a map on second homology $f_*\colon H_2(S) \to H_2(S')$. As $H_2(S) \cong H_2(S') \cong \mathbb{Z}$, we can think of this map as multiplication by an integer d, known as the degree of f.

    (a) Construct maps of degree zero, one and two from a closed orientable genus 3 surface to a closed orientable genus 2 surface.

    (b) Show that there is no degree one map from the closed orientable genus 2 surface to the closed orientable genus 3 surface. Hint: think about cup products or use covering spaces.