Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Fall 2017

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers and include statements of any theorems you cite.

Part I

1. The sum of two subsets A and B of $\mathbb{R}^n$ is defined to be $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.

   (a) Prove or disprove: The sum of two compact sets in $\mathbb{R}^n$ is compact.
   (b) Prove or disprove: The sum of two closed sets in $\mathbb{R}^n$ is closed.

2. Let τ be the standard topology on the unit interval $I = [0, 1]$ and let τ' be another topology on I.

   (a) Prove that if $\tau' \subsetneq \tau$ then I cannot be Hausdorff with the topology τ'.
   (b) Prove that if $\tau \subsetneq \tau'$ then I cannot be compact with the topology τ'.

3. Let Top_* be the category of pointed topological spaces, let $\Sigma \colon \mathsf{Top}_* \to \mathsf{Top}_*$ be the functor that sends a pointed space X to its reduced suspension $\Sigma X := S^1 \wedge X$, and let $\Omega \colon\colon \mathsf{Top}_* \to \mathsf{Top}_*$ be the functor that sends a pointed space X to its based loop space $\Omega X := \hom_{\mathsf{Top}_*}(S^1, X)$. The setup

$$\Sigma \colon \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* \colon \Omega$$

   defines an adjunction. Your problem: described the unit and counit of this adjunction.

4. Let X be the space of continuous function $[0, 1] \to [0, 1]$ with the compact-open topology.

   (a) Is the function $ev \colon X \times [0, 1] \to [0, 1]$ defined by $(f, x) \mapsto f(x)$ continuous?
   (b) Is the map $I \colon X \to [0, 1]$ defined by $f \mapsto \int_0^1 f$ continuous?

Part II

5. Let $G = F(a, b)$ be the free group on two generators. Find all index two subgroups of G up to conjugacy.

6. Below is a picture of a covering $\tilde{X} \xrightarrow{p} X$ of $X = S^1 \vee S^1$.
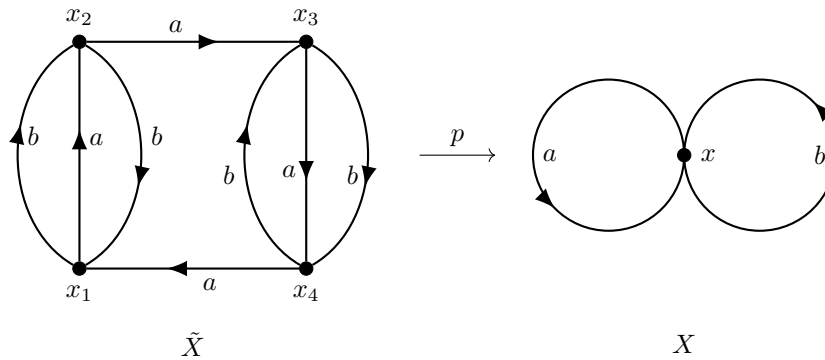
   (a) Compute $\pi_1(\tilde{X}, x_3)$ as a subgroup of $\pi_1(X, x) = F(a, b)$.
   (b) Describe the actions of $a \in \pi_1(X, x)$ and $b \in \pi_1(X, x)$ on the fiber $p^{-1}(x)$.

7. Let X_i be a circle with basepoint x_i, and let α_i be a generator of $\pi_1(X_i)$. Let Y be $X_1 \vee X_2$ with a Möbius band attached along the path $\alpha_1 \cdot \alpha_2$. Use the van Kampen Theorem to write down a presentation for $\pi_1(Y, y_0)$, where the base point y_0 is the identified point $x_1 = x_2$ in Y.

8. Let $X = \{M \in GL_2(\mathbb{C}) \mid M^*M = I\}$ be the set of 2×2 unitary matrices. Compute the homotopy groups $\pi_i(X, x_0)$ with the base point $x_0 = I$ for $i = 0, 1, 2, 3$.

9. Let $T = S^1 \times S^1$ be the torus.

    (a) Show that every finite covering space of T is homeomorphic to T.

    (b) Can any infinite cover of T be homeomorphic to T ?

    (c) Find two non-isomorphic four-fold covers of T by T.

Part III

10. Let X be a compact, connected surface of genus $g \geq 1$.

    (a) Prove that X cannot be covered by two contractible open sets.

    (b) Prove that X can be covered by three contractible open sets.

11. Is $S^2 \vee S^4$ homotopic to a manifold?

12. Let C and D be chain complexes of R modules and let $f \colon C \to D$ be a chain map.

    (a) Define what it means for f to be a homotopy equivalence of chain complexes.

    (b) Prove that if f is a homotopy equivalence then f induces an isomorphism in homology $f_* \colon H(C) \to H(D)$.

13. Let T denote the torus and K denote the Klein bottle.

    (a) Prove that for any map $f \colon T \to K$, the map $f^* \colon H^2(K; \mathbb{Z}_2) \to H^2(T; \mathbb{Z}_2)$ is trivial.

    (b) Using the cup product structure on T, show that for any non-zero $\alpha \in H^1(T)$ there exists $\beta \in H^1(T)$ such that $\alpha \cup \beta \neq 0$.

14. Let X be a finite cell complex with $\chi(X) = \sum(-1)^i \mathrm{rank}(\mathrm{H_i}(\mathrm{X}))$.

    (a) Prove that $\chi(X) = \sum(-1)^i \mathrm{rank}(\mathrm{H^i}(\mathrm{X}))$.

    (b) If X is an odd dimensional orientable manifold, prove that $\chi(X) = 0$.