Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Spring 2017

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers and include statements of any theorems you cite.

Part I

1. (a) Let A be a connected subspace of X, and let $A \subset B \subset \bar{A}$. Show that B is connected.
   (b) Show that the product of arbitrarily many path-connected spaces is path-connected.
2. Let D be a dense subset of X, let Y be Hausdorff, and let $f, g \colon X \to Y$ be maps.
   Show that if $f|_D = g|_D$, then $f = g$.
3. Let $X = \text{spec}(\mathbb{Z})$ denote the set of prime ideals in the ring $\mathbb{Z}$ of integers. The *Zariski topology* on $\text{spec}(\mathbb{Z})$ whose closed sets are sets of the form $V(I) = \{p \in \text{spec}(\mathbb{Z}) : \text{I} \subset \text{p}\}$ for some ideal $I \subset \mathbb{Z}$.
   (a) Prove that $\text{spec}(\mathbb{Z})$ is not Hausdorff.
   (b) Prove that $\text{spec}(\mathbb{Z})$ is compact.
4. Let Top_* be the category of pointed topological spaces, let $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$ be the functor that sends a pointed space X to its reduced suspension $\Sigma X := S^1 \wedge X$, and let $\Omega : \mathsf{Top}_* \to \mathsf{Top}_*$ be the functor that sends a pointed space X to its based loop space $\Omega X := \hom_{\mathsf{Top}_*}(S^1, X)$. The setup

$$\Sigma : \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* : \Omega$$

   defines an adjunction. Your problem: described the unit and counit of this adjunction.

Part II

5. Let X be the two-dimensional torus with one point removed and let Y be the two-dimensional sphere with three points removed.
   (a) Are X and Y homotopic ?
   (b) Are X and Y homeomorphic?
6. Let M_g be the closed orientable surface of genus g.
   (a) Prove that there is no two sheeted cover $M_g \to M_g$ if $g \neq 1$.
   (b) Prove that there are more than one non-isomorphic two sheeted covers $M_1 \to M_1$.
7. Let X be the cell complex obtained by attaching two 2-cells to S^1 by the attaching maps $z \to z^4$ and $z \to z^7$.
   (a) Compute $\pi_1(X)$.
   (b) Is X homeomorphic to S^2 ?
8. (a) Let X be a cell complex, and let $X^{(k)}$ denote its k-skeleton. Prove that $\pi_1(X) = \pi_1(X^{(2)})$.
   (b) Using a cell structure for $\mathbb{RP}^n$ compute $\pi_1(\mathbb{RP}^n)$.
9. Let X be the space obtained by attaching an annulus to the torus, such that the annulus bounds two parallel longitudes (which are pt. $\times S^1$) i.e $X = \Theta \times S^1$. Using Seifert-Van-Kampen theorem compute $\pi_1(X)$.

Part III

10. Let X be a path connected space with $H^1(X, \mathbb{Q}) \neq 0$.
    (a) Prove that $H_1(X)$ has a subgroup isomorphic to $\mathbb{Z}$.
    (b) Does $\pi_1(X)$ have a subgroup isomorphic to $\mathbb{Z}$?

11. Let α, β be two curves on a closed orientable surface S that intersect in one point.
    (a) Prove that α and β each represent non-zero homology classes in $H_1(S)$.
    (b) Suppose instead α, β intersect in two points, could they each represent non-zero homology classes ?

12. Let X be the space obtained by gluing two copies of the solid torus $V = D^2 \times S^1$ along the boundary torus by the identity map. Using Mayer-Vietoris sequence compute the homology of X.

13. Let $n > m$ and suppose $f : \mathbb{CP}^n \to \mathbb{CP}^m$. Prove that the induced map $H^2(\mathbb{CP}^m; \mathbb{Z}) \to H^2(\mathbb{CP}^n; \mathbb{Z})$ is trivial.

14. Let S be the singular chain complex of a point with coefficients in $\mathbb{Z}$. Prove that for any chain complex C of abelian groups, $H_*(C \otimes_{\mathbb{Z}} S) \cong H_*(C)$.