Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Fall 2015

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and include statements of any theorems you cite.

Part I

1. Let X be the subset of $\mathbb{R}^2$ defined by $X = \bigcup_{n=1}^{\infty} S_n$, where $S_n = \{(x, y) \mid (x-1/n)^2 + y^2 = 1/n^2\}$, with the subspace topology, sometimes known as the Hawaiian earring. Show that X is not homeomorphic to a countable wedge of circles.

2. Let τ be the standard topology on the unit interval $I = [0, 1]$ and let τ' be another topology on I.
   (a) Prove that if $\tau' \subsetneq \tau$ then I cannot be Hausdorff with the topology τ'.
   (b) Prove that if $\tau \subsetneq \tau'$ then I cannot be compact with the topology τ'.

3. Consider the rationals $\mathbb{Q} \subset \mathbb{R}$ with the usual subspace topology.
   (a) Show that $\mathbb{Q}$ is not locally compact.
   (b) Show that the one-point compactification $\widehat{\mathbb{Q}}$ is not Hausdorff.

4. Let **Top** be the category of topological spaces, let **CH** be the subcategory of compact Hausdorff spaces and let $\mathbf{U} : \mathbf{CH} \to \mathbf{Top}$ be the functor that is the identity on objects and morphisms. The functor $\mathbf{U}$ has a left adjoint $\beta : \mathbf{Top} \to \mathbf{CH}$ (called the Stone-Čech compactification). Prove that any compact Hausdorff space X is a retract of $\beta\mathbf{U}X$.

Part II

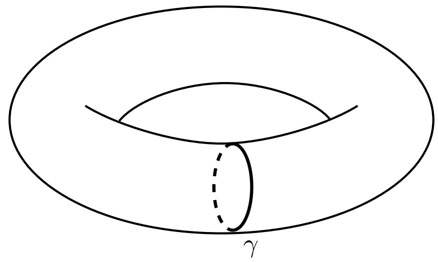
5. Let X be a path-connected, locally path-connected space with finite fundamental group. Let $Y = S^1 \times S^1 \times \cdots \times S^1$, a product of n copies of the circle with $n \geq 1$. Show that every map $f\colon X \to Y$ is null-homotopic.

6. Let $X = \mathbb{RP}^2 \vee S^1$. Carefully enumerate all connected 3–fold covers of X.

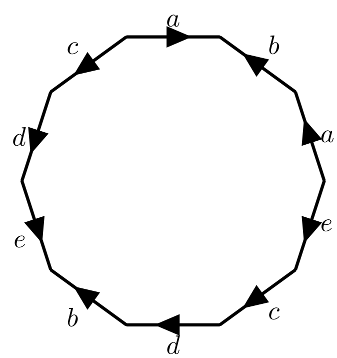
7. Let T_n be the surface obtained by removing $n > 1$ points from the torus. How many isomorphism classes of regular, connected, 2-fold covers of T_n are there?

8. Let X be the union of the unit 2-sphere in $\mathbb{R}^3$, together with the segment of the z-axis inside the 2-sphere (with the subspace topology).
   (a) Compute $\pi_1 X$.
   (b) Describe the universal cover $\widetilde{X}$, including a local picture, and describe the action of $\pi_1 X$ on $\widetilde{X}$.
   (c) Show that X has an n-fold cover for each $n \in \mathbb{N}$.

9. The space X is obtained by attaching a Möbius strip along its boundary to one of the meridians of a two-torus (marked γ in the picture below). Find the fundamental group of X for some choice of a base point.



10. State the classification of closed orientable surfaces. Compute the Euler characteristic of the surface obtained by identifying the sides of the polygon as indicated below, and identify which surface it is.



Part III

11. Consider the operation on surfaces of self-gluing. For a surface M, remove the interior of two discs and glue along the boundaries. Notice that there are two ways of doing this: an orientation preserving and an orientation reversing. Denote the resulting surfaces by M^+ and M^-. Determine M^+ and M^- for: the sphere, the torus, and the projective plane.

12. Let $X = S^1 \vee S^1 \vee S^2$.

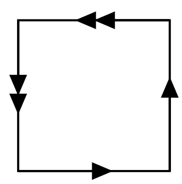
    (a) Show that X and the 2-torus T^2 have CW-structures with four cells: one 0-cell, two 1-cells and one 2-cell.

    (b) Use cellular homology to show that T^2 and X have isomorphic homology groups.

    (c) Use cup products to show that T^2 and X are not homeomorphic.

13. Caclulate the cohomology ring of $\mathbb{CP}^2$. Then calculate the cohomology ring of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

14. Consider a surface obtained by identifying edges of a square as indicated below.



    (a) Give this space a Δ-complex structure and use it to compute its simplicial homology.

    (b) Let point p be at the center of the above square and D a small open two-disk centered at p. Compute the homology of the surface via the Mayer-Vietoris sequence, using D and the complement of p as the two open sets.

15. Let M be a compact orientable 3-manifold with non-empty boundary. Show that the kernel of the map $i_*\colon H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is non-trivial, where $i\colon \partial M \to M$ is the inclusion map.