Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Fall 2014

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers and include statements of any theorems you cite.

Part I

1. Show that if X is a separable metric space, i.e. it has a countable dense subset, then the topology on X is second countable.

2. (a) Show that a contractible space is path connected.
   (b) Show that every path connected space is connected.

3. (a) Show that a closed subspace of a compact space is compact.
   (b) Show that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Part II

4. Let T be the compact surface with boundary formed by removing a small open disc from a torus. Find all connected two-fold covers of T, up to covering space isomorphism, carefully justifying your answer. Let $f\colon S^1 \to \partial T$ be a homeomorphism from a circle to the boundary of the surface T. For which of the covers T, if any, is there a lift $f\colon S^1 \to T$?

5. Show directly that every map from S^1 to S^2 is null-homotopic. Show that every map from S^2 to S^1 is null-homotopic.

6. Compute the fundamental group of $\mathbb{R}^3$ with the unit circle in the xy-plane and the z-axis removed. Note that this space is homeomorphic to the complement of the Hopf link (two linked unknotted circles) in S^3. Please justify all your steps.

7. Let X be a connected finite CW-complex with one 0-cell, two 1-cells, three 2-cells, and some 3-cells. Suppose the three 2-cells are attached to $X^{(1)} = S^1 \vee S^1$, the 1-skeleton of X, by (based) maps $g_i : S^1 \to S^1 \vee S^1$, such that $[g_i] = w_i(a,b) \in \pi_1(X^{(1)})$ for $i = 1,2,3$. Find necessary and sufficient conditions on $w_i(a,b)$ so that $H_1(X,\mathbb{Z}) = 0$.

8. Let X be the wedge of two circles $S^1 \vee S^1$, with basepoint x_0 the common point on the two circles, let a and b be two loops based at x_0, such that each loop goes exactly once around one of the circles, and they generate $\pi_1 X$. Let $f\colon X \to X$ be a map which sends $a \mapsto b$ and $b \mapsto bab^{-1}$. Use van Kampen's theorem to find a presentation for the fundamental group of the mapping torus of f i.e the space $X_f = X \times [0,1]/(x \sim f(x))$.

Part III

9. Let T be the compact manifold with boundary obtained from removing a small open disc from a torus $S^1 \times S^1$. Use the Mayer-Vietoris theorem to compute the homology of the space obtained by gluing a Möbius band to T by a homeomorphism identifying their boundaries.

10. Let S and S' be closed orientable surfaces. A map $f\colon S \to S'$ induces a map on second homology $f_*\colon H_2(S) \to H_2(S')$. As $H_2(S) \cong H_2(S') \cong \mathbb{Z}$, we can think of this map as multiplication by an integer d, known as the degree of f.

    (a) Construct maps of degree zero, one and two from a closed orientable genus 3 surface to a closed orientable genus 2 surface.

    (b) Show that there is no degree one map from the closed orientable genus 2 surface to the closed orientable genus 3 surface. Hint: think about cup products or use covering spaces.

11. Let X be a closed surface obtained by identifying the sides of an octagon using the word $aba^{-1}cd^{-1}b^{-1}dc$.

    (a) Describe the CW-complex structure on X given by this description.

    (b) Compute the cellular chain complex and use it to compute the homology of X.

    (c) Identify the surface X.

12. Prove of disprove: Any space X with homology $H_i(X) = \mathbb{Z}$ for $i = 0, 1, 2, 3$ is homeomorphic to $S^1 \times S^2$.

13. Let X be a CW-complex obtained from S^1 by attaching two 2-cells by maps of degrees p and q respectively, where p and q are relatively prime.

    (a) Show that X is simply connected.

    (b) Compute the integral homology groups of X.

    (c) Is X homeomorphic to S^2 ? Is X homotopy equivalent to S^2 ?