Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Spring 2014

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers and include statements of any theorems you cite.

Part I

1. Let be X a metric space with a countable number of points. Then X is connected if and only if X consists of at most one point.

2. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ endowed with the subspace topology inherited from $\mathbb{R}^2$. Let X be topological space. Prove that if there exists a continuous bijection $f\colon D \to X$, then X is homeomorphic to D.

3. Prove or disprove: $\mathbb{Q}$ is homeomorphic to $\mathbb{N}$, with their usual topologies.

Part II

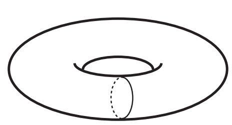
4. Find all connected three-fold covers of the wedge sum of two projective planes, carefully justifying your answer.

5. Let M and N be closed, connected and orientable n-manifolds. Let $f\colon M \to N$ be a degree one map. Show that $f_*\colon \pi_1(M) \to \pi_1(N)$ is onto.

6. Let $X \subset \mathbb{R}^3$ denote the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

7. One way to represent a topological surface is as a polygon with pairs of sides identified in some fashion. Describe two topologically different surfaces that can be made in this way from a hexagon, identify which surfaces you obtained, and prove that these are topologically distinct.

8. Let γ be a closed curve on the torus as pictured. Let S be the topological space formed by gluing a möbius strip to the torus by identifying γ with the boundary of the möbius strip. Compute the fundamental group of S.



Part III

9. Write down an explicit cell structure for the space obtained by taking two copies of $\mathbb{R}P^2$ and identifying copies of $\mathbb{R}P^1$ in each one by a homeomorphism, and use cellular homology to compute its homology groups.

10. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same cohomology groups in each dimension. Describe the ring structure on cohomology, and deduce that they are not homotopy equivalent.

11. Use the Mayer-Vietoris theorem to compute the homology of the space obtained by gluing two Möbius bands together by a homeomorphism identifying their boundaries.

12. What can you say about the homology of a closed orientable 3-manifold with trivial fundamental group?

13. Let M be a compact orientable 3-manifold with non-empty boundary. Show that the kernel of the map $i_*\colon H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is non-trivial, where $i\colon \partial M \to M$ is the inclusion map.