Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Fall 2013

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers.

Part I

1. Characterize the continuous maps from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with its usual topology.

2. Prove that if a topological space is path connected, then it is connected.

3. Let X and Y be topological spaces and $f\colon X \to Y$ be a continuous bijection.

   (a) Prove that if Y is Hausdorff then X is Hausdorff.

   (b) If X is Hausdorff, does Y need to be Hausdorff? Explain.

Part II

1. State the classification of closed orientable surfaces. Compute the Euler characteristic of the surface obtained by identifying opposite sides of a 10-gon with opposite orientation (i.e. according to the identifications $abcdeABCDE$), and identify which surface it is.

2. Find all connected three-fold covers of the wedge sum of a circle with a projective plane, carefully justifying your answer.

3. Show that every map from S^2 to S^1 is null homotopic.

4. Find the fundamental group of the space obtained by attaching a disc to a torus by a map which is a homeomorphism form the boundary of the disc to an essential simple closed curve on the torus. Show that the resulting space is not homotopy equivalent to a circle.

5. Compute the fundamental group of $\mathbb{R}^3$ with two circles of radius one removed, one centered at $(2, 0, 0)$ and the other at $(-2, 0, 0)$ (i.e. an unlink of two components).

Part III

1. Let X be a finite CW-complex, and let $X^{(k)}$ denote its k-skeleton.

   (a) Prove that $H_*(X^{(k)}, X^{(k-1)}) \cong \oplus_{e_i^k} H_*(S^k)$, where e_i^k denotes the k-cells of X.

(b) Prove that the inclusion $i\colon X^{(k)} \to X$ induces an isomorphism $i_*\colon H_i(X^{(k)}) \to H_i(X)$ for $i < k$.

(c) Give an example to show that i_* need not be an isomorphism for $i = k$.

2. Show that $\mathbb{CP}^2$ and $S^2 \vee S^4$ have the same cohomology groups in each dimension. Describe the ring structure on cohomology, and deduce that they are not homotopy equivalent.

3. Use the Mayer-Vietoris theorem to compute the homology of $S^2 \widetilde{\times} S^1$, i.e. the space obtained from $S^2 \times I$ by identifying $S^2 \times \{0\}$ and $S^2 \times \{1\}$ by the antipodal map. (Hint: consider the space as the union of two copies of $S^2 \times I$, with one pair of ends glued together by the identity map, and the other pair by the antipodal map.)

4. Show that a compact manifold with non-empty boundary does not retract onto its boundary.

5. Show that if $H_k(X;\mathbb{Z})$ is finitely generated and free for all k then $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings.