Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

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Instructions: Do at least 8 problems in all, with exactly two problems from Part I and at least two problems from Parts II & III each.

Part I

1. Let X and Y be topological spaces and $f : X \to Y$ be a continuous surjection.

   (a) Prove that if X is connected then Y is connected.

   (b) If Y is connected, does X need to be connected ? Explain.

2. Let X and Y be topological spaces and $f : X \to Y$ be a continuous bijection.

   (a) Prove that if Y is Hausdorff then X is Hausdorff.

   (b) If X is Hausdorff, does Y need to be Hausdorff ? Explain.

3. Let f be a continuous, real-valued function defined on non-empty, compact, connected space. Prove that the image of f is a closed interval.

4. (a) Show that a compact subset of a Hausdorff space is closed.

   (b) Show that a contractible space is path connected.

Please Turn Over

Part II

1. (a) Let X and Y be topological spaces with basepoints $x \in X$ and $y \in Y$ respectively. Prove that $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$.

   (b) Compute $\pi_1(T^2 \times \mathbb{R}\mathrm{P}^n, (x, y))$.

2. Classify all covering spaces of $S^1 \times \mathbb{R}\mathrm{P}^2$ for $n \geq 2$.

3. Use a covering space argument to prove that any map $f : \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2 \to S^1$ is null homotopic.

4. Let $Y \subset \mathbb{R}^3$ be the union of the three co-ordinate axes and let x be a point not on Y. Compute $\pi_1(\mathbb{R}^3 - Y, x)$.

5. Let N_g denote the closed non-orientable surface of genus g. Compute $\pi_1(N_g, x)$.

6. Let X be a CW complex with one 0-cell, three 1-cells a, b and c and two 2-cells e and f, with attaching maps given by the words ab^2c and cab^{-3} respectively. Compute $\pi_1(X, x)$ with some base-point x.

Please Turn Over

Part III

1. Let X be a closed surface given by the identification of sides of an octagon using the word $abca^{-1}dc^{-1}db$.

   (a) Describe the CW structure on X given by this description.

   (b) Compute the cellular chain complex and use it to compute $H_*(X;\mathbb{Z})$.

   (c) Identify the surface.

2. Compute $H_*(\mathbb{R}\mathrm{P}^4;\mathbb{Z})$.

3. Let $X = S^2 \vee S^4 \vee S^6$ and $Y = \mathbb{C}\mathrm{P}^3$.

   (a) Compute the homology and cohomology groups of X and Y (you can use known results) with $\mathbb{Z}$ coefficients. Are they the same ?

   (b) Are the spaces homeomorphic ? Are the spaces homotopy equivalent ? Explain.

4. Use Poincare duality to show that any odd dimensional manifold has zero Euler characteristics (Hint: Treat the non-orientable case differently).

5. Let X be a connected space with the following homology groups with coefficients in $\mathbb{Z}$: $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}\oplus\mathbb{Z}_3$, $H_2(X) = \mathbb{Z}_4\oplus\mathbb{Z}_5$, $H_3(X) = \mathbb{Z}\oplus\mathbb{Z}_2$, and $H_n(X) = 0$ for $n \geq 4$.

   (a) Compute $H_*(X;\mathbb{Z}_3), H_*(X;\mathbb{Z}_4), H_*(X,\mathbb{Q})$.

   (b) Compute $H^*(X;\mathbb{Z}_2), H^*(X;\mathbb{Z}_5), H^*(X,\mathbb{Q})$.

6. (a) Let (X, x) and (Y, y) be good pairs and $X \vee Y$ be the wedge sum obtaining by identifying the points x and y. Prove that $\widetilde{H}_k(X \vee Y) \simeq \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y)$ for all $k \geq 0$.

   (b) Let X be a CW complex which has five 4-cells. Let $X^{(n)}$ denote the n–skeleton of X. Compute $H_*(X^{(4)}, X^{(3)};\mathbb{Z})$.