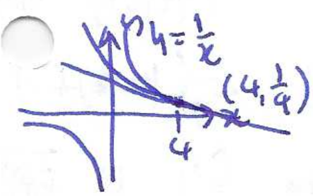
so equation of tangent line is $y-1 = 2(x-1)$ $\quad y = 1+2(x-1)$

$y = 2x-1$

Example find the tangent line to $y=\frac{1}{x}$ at $x=4$



slope $f'(4) = \lim_{h\to 0} \dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \dfrac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{4+h} - \frac{1}{4}\right) = \lim_{h\to 0} \dfrac{4-(4+h)}{h(4+h)4} = \lim_{h\to 0} \dfrac{-h}{h(4+h)4}$

$= \lim_{h\to 0} \dfrac{-1}{4(4+h)} = -\frac{1}{16}$ $\quad$ tangent line: $y - \frac{1}{4} = -\frac{1}{16}(x-4)$

Example: find slope of tangent line to $f(x) = mx+b$ at $x=a$:

$f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{m(a+h)+b-(ma+b)}{h} = \lim_{h\to 0} \dfrac{mh}{h}$

$= \lim_{h\to 0} m = m.$ $\quad$ observation: if $f(x) = b$ (constant)

then $f'(x) = 0$ for all x

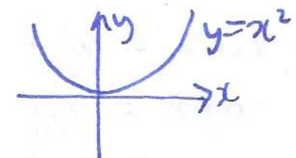
§3.2 Derivative as a function

at each point x, there is a tangent line, with a slope, which is a number. Therefore we can define a function $\mathbb{R} \to \mathbb{R}$

$x \mapsto$ slope of tangent line at $(x, f(x))$

notation: we call this function $f'(x)$ "the derivative of f"

Example

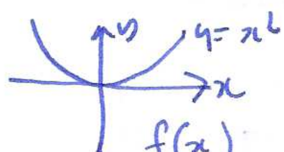


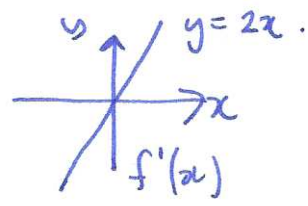
if $f(x) = x^2$, then $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \dfrac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h$

$= \cancel{\lim}\ 2x.$ $\quad$ Summary: if $f(x) = x^2$

then $f'(x) = 2x$

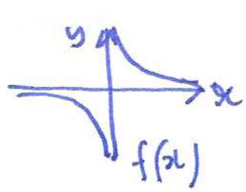


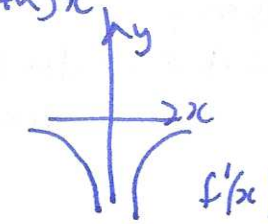


Example $f(x) = \frac{1}{x}$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \dfrac{x-(x+h)}{h(x+h)x}$

$= \lim_{h\to 0} \dfrac{-h}{h(x+h)x} = \lim_{h\to 0} \dfrac{-1}{(x+h)x} = -\dfrac{1}{x^2}$





<u>Remarks</u> ① functions f: domain $\to$ range eg. $f: \mathbb{R} \to \mathbb{R}$

derivative : (differentiable) functions $\to$ functions

$$f \longmapsto f'$$

warning: not all functions differentiable.

② "calculus" means rules for doing calculations. We won't have to explicitly compute limits all the time.

<u>Example</u> $f(x) = x^3$ $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim 3x^2+3xh+3h^2 = 3x^2$ $\quad$ Q: what about x^{100}?

<u>Thm</u> (powers of x) If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ (works for all $n \in \mathbb{R}$!)

<u>Examples</u> $\frac{d}{dx}(x^2) = 2x$ $\quad \frac{d}{dx}(x^3) = 3x^2$ $\quad \frac{d}{dx}(x) = 1$ $\quad \frac{d}{dx}(1) = 0$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$ $\quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

<u>Thm</u> $\frac{d}{dx}(x^n) = nx^{n-1}$

<u>Proof</u> $f(x) = x^n$ $\quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

<u>Fact</u>: binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + h^n$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\quad$ so $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + h^2(-) - x^n}{h}$

$= \lim_{h\to 0} nx^{n-1} + h(-) = nx^{n-1}$ $\square$

<u>warning</u>: this rule only works for polynomials, <u>not</u> exponentials

$f(x) = x^{100}$ $\quad f'(x) = 100x^{99}$ $\quad f(x) = 2^x$ <u>not</u> polynomial.

<u>other useful rules</u> <u>Thm</u> (linearity) If f and g are differentiable functions, then:

- $f+g$ is differentiable, with $(f+g)' = f'+g'$

$\Leftrightarrow \quad \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

- k constant: $(kf)' = kf' \Leftrightarrow \frac{d}{dx}(kf) = k\frac{df}{dx}$

Proof (follows from limit laws)

- $(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$

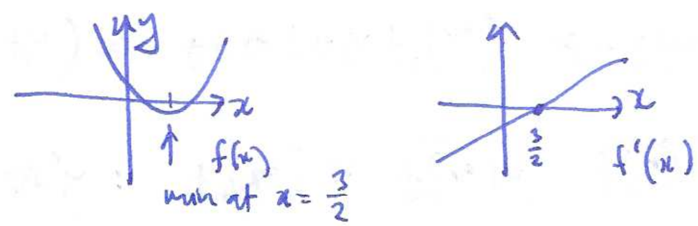
- $(kf)'(x) = \lim_{h\to 0} \frac{(kf)(x+h)-(kf)(x)}{h} = \lim_{h\to 0} \frac{kf(x+h)-kf(x)}{h}$

$= \lim_{h\to 0} k\frac{f(x+h)-f(x)}{h} = k\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = kf'(x) \quad \square$

Example $f(x) = x^2 - 3x+2$. Find $f'(x)$

$\frac{df}{dx} = \frac{d}{dx}(x^2-3x+2) = \frac{d}{dx}(x^2) - 3\frac{d}{dx}(x) + \frac{d}{dx}(2) = 2x-3+0 = 2x-3.$

graphs:



Derivative of e^x — consider $f(x) = b^x \quad (b>0)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} b^x \frac{b^h-1}{h}$

$= b^x \underbrace{\lim_{h\to 0} \frac{b^h-1}{h}}$ → this limit does not depend on x!
Assume this limit exists and call it m_b

We have shown: for exponential functions, the derivative is proportional to the original function, i.e. if $f(x) = b^x$ then $f'(x) = m_b b^x$. in particular, the slope at $x=0$ is m_b.