~~Example~~ 0-length interval: $\int_a^a f(x)\,dx = 0$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

comparisons: $f(x) \leq g(x) \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

§5.3 Antiderivatives

Def$^{\underline{n}}$ A function $F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$

Example $f(x) = x^2$, $F(x) = \frac{1}{3}x^3$. Note: $\frac{1}{3}x^3+1$ also works.

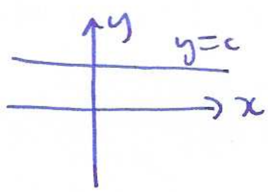
General antiderivative

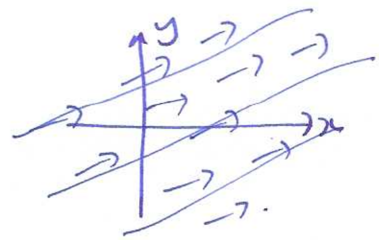
Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$, for some $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$. then $F'(x) = G'(x) = f(x)$. so $(F(x)-G(x))' = f(x) - f(x) = 0$, so $F(x) - G(x) = c$ const. □

Picture $f(x)$ gives the slope function for $F(x)$.

Example $f(x) = c$





$f(x) = c$

$F(x) = cx+d$

Notation $\int f(x)\,dx$ means the general antiderivative $F(x)+c$

Example $\int x^2 + \frac{1}{x} + \sin x\,dx = \frac{1}{3}x^3 + \ln|x| - \cos x + C$

Observation: every rule for differentiation gives a rule for integration.

Warning: no simple analog of product/quotient/chain rule :(

Alternate view: we can think of finding the indefinite integral as finding a function given its slope, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general, there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (called an initial condition) then this picks out a particular solution.

Example motion under gravity acceleration: $a(t) = x''(t) = -g$ (constant)

velocity: $v(t) = x'(t) = -gt + c$

If $v(0) = v_0$, initial velocity, then $v(t) = -gt + v_0$

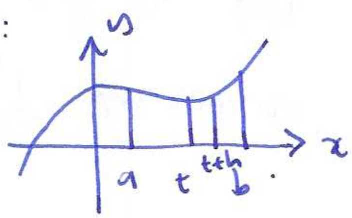
position: $x(t) = -\frac{1}{2} g t^2 + v_0 t + c$

if $x(0) = x_0$, initial position, then $x(t) = -\frac{1}{2} g t^2 + v_0 t + x_0$

§5.4 Fundamental theorem of calculus I

Thm (FTC ①) Suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

intuition:



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt t?

$$\frac{d}{dt} \int_a^t f(x)\,dx = \lim_{h \to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(x)\,dx$$

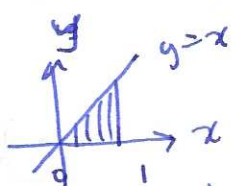
$$\approx \frac{\text{area of rectangle}}{h} = \frac{f(t) \times h}{h} = f(t).$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x)\,dx = F(t) + c$

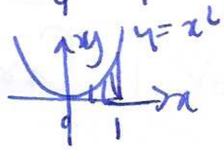
Q: what is the constant? $t = a$: $\int_a^a f(x)\,dx = 0 = F(a) + c \Rightarrow c = -F(a)$

so $\int_a^t f(x)\,dx = F(t) - F(a)$ □

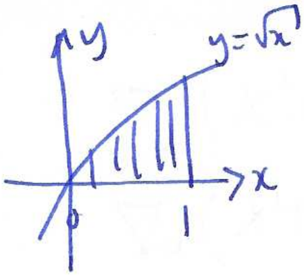
Examples



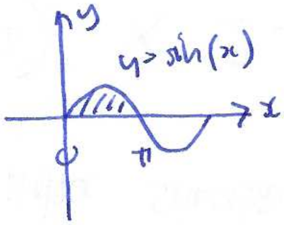
$$\int_0^1 x\,dx = \left[\tfrac{1}{2} x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$



$$\int_0^1 x^2\,dx \quad \left[\tfrac{1}{3} x^3\right]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3}$$

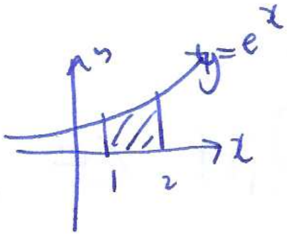


$$\int_0^1 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

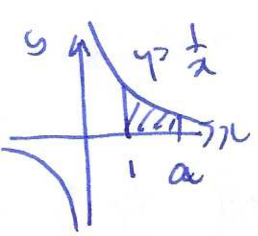


$$\int_0^{\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{\pi} = -\cos(\pi) + \cos(0) = 1+1 = 2$$

$$\int_0^{2\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{2\pi} = -(+1)+1 = 0.$$



$$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2 - e$$



$$\int_1^a \frac{1}{x}\,dx = \left[\ln|x|\right]_1^a = \ln(a) - \ln(1) = \ln(a).$$

<u>Observations</u>

① choice of antiderivative doesn't matter, let $F(x)$ and $F(x)+c$ be antiderivatives for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

$$= (F(b)+c) - (F(a)+c) = F(b) - F(a).$$

② $\int_a^t f(x)\,dx$ is a function of t! (and a, f) but <u>not</u> x (x is called a dummy variable or bound variable) so $\int_a^t f(x)\,dx = \int_a^t f(y)\,dy$. If you want a function of x write $\int_a^x f(t)\,dt$.

<u>§5.4 Fundamental theorem of calculus II</u>

<u>Thm</u> (FTC ②) Let $f(x)$ be a continuous function on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$, i.e. $A'(x) = f(x) = \frac{dA}{dx}$, so $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Furthermore $A(a) = 0$ □.