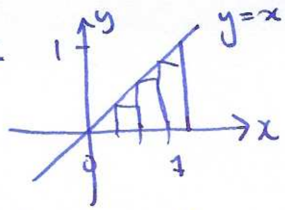
Example



find area under graph of $y=x$ between $x=0$ and $x=1$
(answer $= \frac{1}{2}$)

approximate by 4 rectangles : area $\approx$ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$$\frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{2}{4}\right) + f\left(\frac{3}{4}\right) \right) = \frac{1}{4} \sum_{i=0}^{3} f\left(\frac{i}{4}\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) = \frac{1}{16}(0+1+2+3) = \frac{6}{16} = \frac{3}{8} = 0.375$$

approximate with n rectangles :

$$\frac{1}{n}f(0)\frac{1}{n} + \frac{1}{n}f\left(\frac{1}{n}\right) + \frac{1}{n}f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n}f\left(\frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i}{n}$$

$$= \frac{1}{n^2}\left(0+1+2+\cdots+(n-1)\right) = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

claim : $1+2+3+\cdots+(n) = \frac{1}{2}n(n+1)$

Proof ① pairing $\underbrace{1+2+3+\cdots \quad +(n-1)+n}_{n+1}$

n even : $\frac{1}{2}n(n+1)$

n odd : $\frac{n-1}{2}(n+1) + \frac{n+1}{2} = \frac{1}{2}n(n+1)$

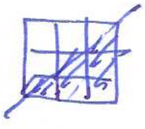
② induction assume true for some k: $S_k = 1+2+\cdots+k = \frac{1}{2}k(k+1)$

consider $S_{k+1} = \underbrace{1+2+\cdots+k}_{S_k}+(k+1) = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{1}{2}k+1\right)$

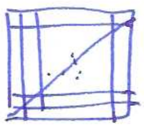
$= \frac{1}{2}(k+1)(k+2) \leftarrow$ ok $\checkmark$ D.

③ picture

$\frac{1}{2}(2)^2 + \frac{1}{2}\cdot 2$



$\frac{1}{2}(3)^2 + \frac{1}{2}\cdot 3$



$\frac{1}{2}(n)^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$

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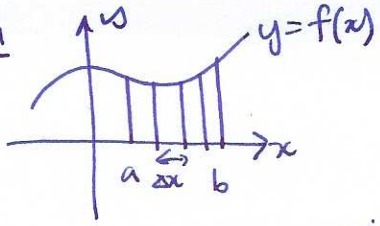
so approximate area with n triangles is $\frac{1}{n^2}\sum_{i=0}^{n-1} i = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{1}{2}\frac{n^2-n}{n^2}$

$= \frac{1}{2}\left(1-\frac{1}{n}\right)$ $\quad \lim_{n\to\infty} \frac{1}{2}\left(1-\frac{1}{n}\right) = \frac{1}{2}$

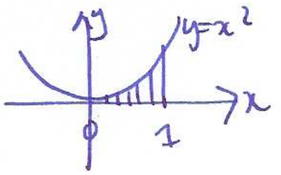
Notation



if N rectangles of equal width $\Delta x = \frac{b-a}{N}$

- left endpoint rectangles : $L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$
- right endpoint rectangles : $R_N = \sum_{i=1}^{N} f(a+i\Delta x)\Delta x$
- midpoint rectangles : $M_N = \sum_{i=1}^{N} f(a+(i-\frac{1}{2})\Delta x)\Delta x$

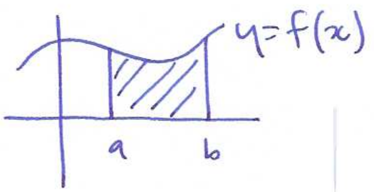
Q: what about $y=x^2$?



need to find $1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

want to find a better way...

§5.2 Definite integral



want : area under the curve $y=f(x)$ between $x=a$ and $x=b$

notation : $\int_a^b f(x)\,dx$

Formal defn : Riemann sum $R(f,P,c)$

P: partition of $[a,b]$

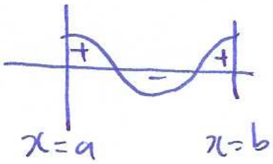
$a = x_0 < x_1 < \dots < x_n = b$

$\Delta x_i = x_i - x_{i-1}$

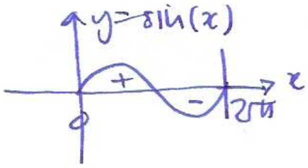
$c_i \in [x_{i-1}, x_i]$

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

note : signed area !

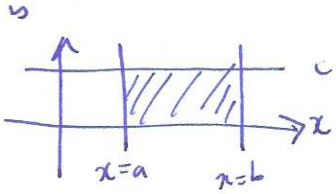


so $\int_0^{2\pi} \sin(x)\,dx = 0$



useful properties

$\int_a^b c\,dx = c(b-a)$



sums : $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

constant multiple : $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

adjacent intervals : $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$