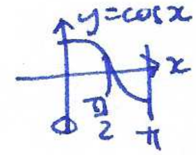
Thm (First derivative test) If $f(x)$ is differentiable and $f'(c)=0$ then if $f'(x)$ changes from $+ve$ to $-ve$ at $c \Rightarrow c$ local max

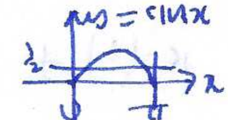
$-ve$ to $+ve$ at $c \Rightarrow c$ local min $\square$.

Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0, \pi]$

- find critical points : $f'(x) = \cancel{\cos^2 x + \sin x}\ -2\cos x \sin x + \cos x$
- solve $f'(x) = 0$ $\quad \cos(x)(1-2\sin x) = 0 \quad$ • $\cos(x) = 0 \quad x = \frac{\pi}{2}$.



$\sin(x) = \frac{1}{2}$



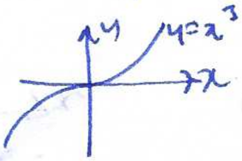
$x = \frac{\pi}{6}, \frac{5\pi}{6}$ so critical points are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

find sign of $f'(x)$:

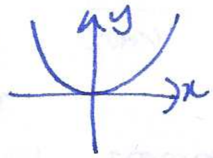
	0	$\pi/6$	$\pi/2$	$5\pi/6$	π
$\cos x$	$+$	$+$	$-$	$-$	
$1-2\sin x$	$+$	$-$	$-$	$+$	
$f'(x)$	$+$	$-$	$+$	$-$	
		local max	local min	local max	

Example critical point not max or min

$f(x) = x^3$



$f'(x) = 3x^2$

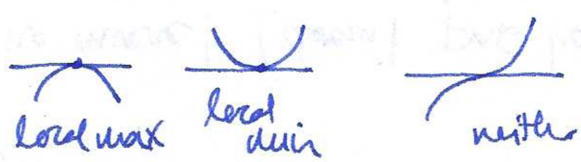


$f'(x) = 0$ at $x = 0$

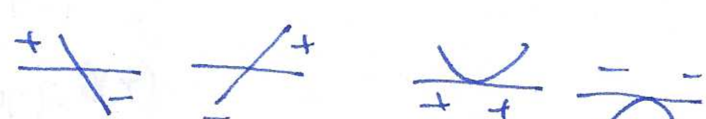
$f'(x)$: $+ve$ $\quad 0 \quad$ $+ve$

$\Rightarrow x = 0$ not local max or min

recall critical point $f'(x) = 0$ $\quad f(x)$



$f'(x)$

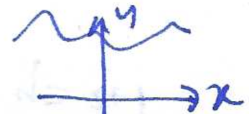


$f''(x)$ $\quad -ve \quad +ve \quad 0 \quad 0$

§4.4 Second derivative test

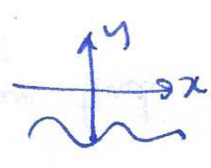
recall $f(x) > 0$

f positive



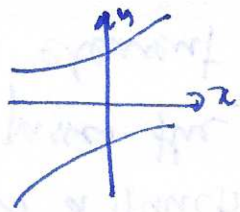
$f(x) < 0$

f negative



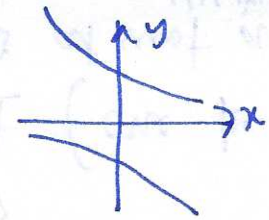
$f'(x) > 0$

f increasing



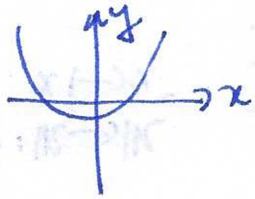
$f'(x) < 0$

f decreasing



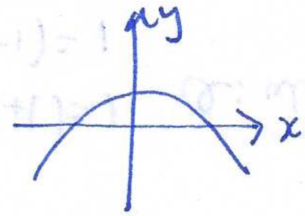
$f''(x) > 0$

f concave up



$f''(x) < 0$

f concave down



mnemonic :

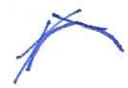
up down.

Q: how do the slopes change?

concave up $\Leftrightarrow$ slopes increasing

$\Leftrightarrow f''(x) > 0$

concave down $\Leftrightarrow$ slopes decreasing

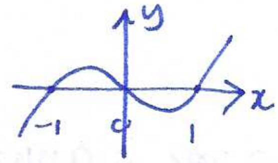


$\Leftrightarrow f''(x) < 0$

Defn Let $f(x)$ be differentiable on an interval (a,b), then
$f(x)$ is concave up on (a,b) iff $f''(x) > 0$
$f(x)$ concave down on (a,b) iff $f''(x) < 0$

Defn An inflection point is a point where the graph changes from concave up to concave down, or vice versa.

Note: inflection point $\Rightarrow f''(x) = 0$
$\not\Leftarrow$



Example $f(x) = x^3 - x = x(x^2-1) = x(x+1)(x-1)$

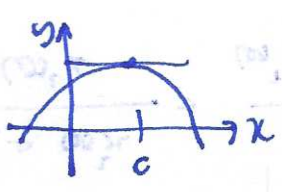
$f'(x) = 3x^2 - 1$

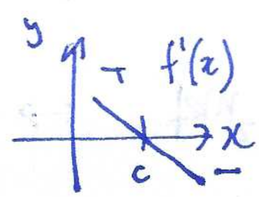
$f''(x) = 6x$

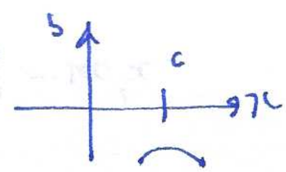
$\left.\begin{array}{ll} f''(x) > 0 & \text{for } x > 0 \\ f''(x) < 0 & \text{for } x < 0 \end{array}\right\}$ $x = 0$ is an inflection point

Second derivative

Local max

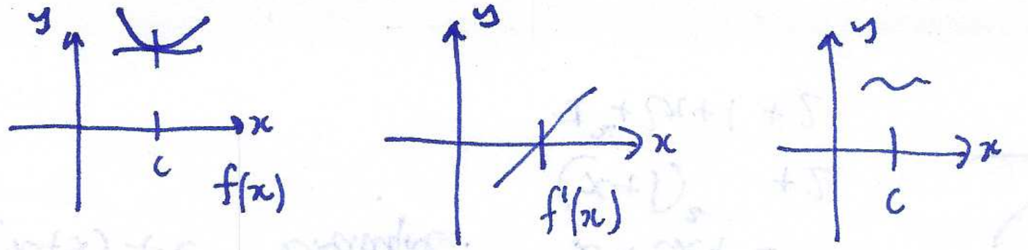






$f''(c) < 0$

local min



$f''(c) > 0$

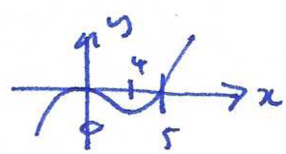
Thm Suppose $f(x)$ is differentiable and c is a critical point,

if $f''(c) > 0 \Rightarrow c$ is a local min

$f''(c) < 0 \Rightarrow c$ is a local max

$f''(c) = 0$ NO INFORMATION (may be local max/min / neither)

Example $f(x) = x^5 - 5x^4 = x^4(x-5)$



$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^3$ critical points $x=0$, $x=4$

2nd derivative test: $x=0$ $f''(0)=0$, no information

$x=4$ $f''(4) = 320 > 0$ local min

at $x=0$ use the first derivative test:

		0	
$(x-4)$	$-$		$-$
x^3	$-$		$+$
$f'(x)$	$+$		$-$

$\Rightarrow$ local max

§4.5 L'Hôpital's rule

Thm Suppose $f(x)$ and $g(x)$ are differentiable and $f(a) = g(a) = 0$
or $f(a) = g(a) = \pm\infty$

then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided this limit exists

warning ① this is not the quotient rule!

② $\dfrac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ not $\frac{1}{0}$, i.e. can't use l'Hôpital on $\displaystyle\lim_{x\to 0} \frac{1}{x}$

Examples ① $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{3x^2}{1} = 3.$