write this as $\displaystyle\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$

set $k = g(x+h) - g(x)$, as g is continuous, $h \to 0 \Rightarrow k \to 0$

$$= \lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(g(x)) . g'(x) . \square .$$

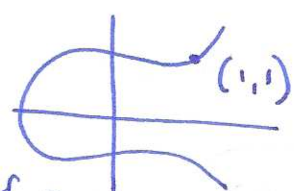
<u>Examples</u> $\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1} . g'(x)$

$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} . g'(x) .$

$\frac{d}{dx}\left(f(ax+b)\right) = a f'(ax+b) .$

<u>§3.8 Implicit differentiation</u>

Consider : $y^4 + xy = x^3 - x + 2$



don't know how to solve for y explicitly

can think of y as a function of x
i.e. $y(x)$:

$(y(x))^4 + x\,y(x) = x^3 - x + 2$ ← differentiate using the chain rule

$4(y(x))^3 . y'(x) + y(x) + x . y'(x) = 3x^2 - 1$

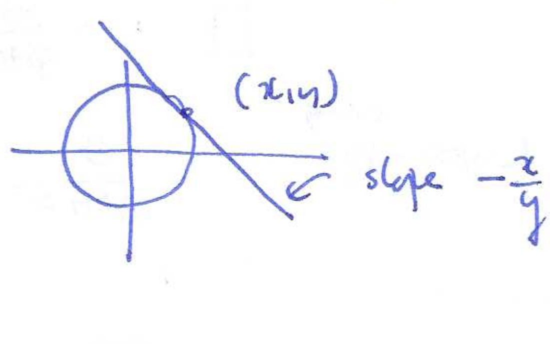
$y'(4y^3 + x) = 3x^2 - 1 - y$

$y' = \dfrac{3x^2 - 1 - y}{4y^3 + x}$ $\qquad y'(1,1) = \frac{1}{5}$

<u>Example</u> $x^2 + y^2 = 1$

$2x + 2yy' = 0$

$y' = -x/y$



Application: derivatives of inverse functions

$$\boxed{\text{Thm } \frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

Example $y = \ln(x)$

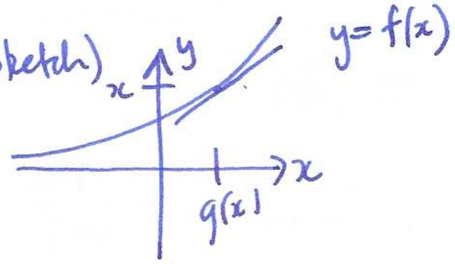
$$e^y = x$$

$$e^y \cdot y' = 1 \qquad y' = e^{-y} = e^{-\ln(x)} = \frac{1}{x}$$

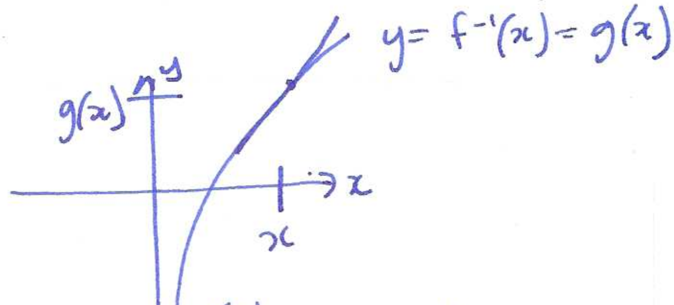
Thm If $f(x)$ is differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

Proof (sketch)



reflect in $y=x$ $\rightarrow$



$\dfrac{1}{\text{slope at } g(x) \text{ of } f(x)} = \text{slope at } g(x) \text{ at } x : g'(x)$

so $g'(x) = \dfrac{1}{f'(g(x))}$ $\square$.

Application derivatives of trig functions

Thm $\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \dfrac{d}{dx}(\cos^{-1}(x)) = \dfrac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$):

$$y = \sin^{-1}(x)$$

$$\sin(y) = x$$

$$\cos(y) \cdot y' = 1$$

$$y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \square$$

Thm $\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{1}{1+x^2}$ $\qquad \dfrac{d}{dx}(\cot^{-1}(x)) = \dfrac{-1}{1+x^2}$

$\dfrac{d}{dx}(\sec^{-1}(x)) = \dfrac{1}{|x|\sqrt{x^2-1}}$ $\qquad \dfrac{d}{dx}(\csc^{-1}(x)) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

Proof (of $\tan^{-1}(x)$)

$$y = \tan^{-1}(x)$$
$$\tan(y) = x$$
$$\sec^2(y)\, y' = 1$$
$$y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \quad \square$$

Example $\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \frac{1}{1+(e^{2x})^2} \cdot e^{2x} \cdot 2$

§3.9 Derivatives of exponentials and Logs

recall: $f(x) = b^x$ then $f'(x) = m_b\, b^x$

$f(x) = e^x$ then $f'(x) = e^x$

Thm $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{x\ln(b)}$

$$f'(x) = e^{x\ln(b)} \cdot \ln(b) = \ln(b)\, b^x \quad \square$$

Example $f(x) = 3^{4x} = (3^4)^x \quad f'(x) = (3^4)^x \cdot \ln(3^4) = 4\ln(3)\, 3^{4x}$

Thm $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$ $\square$

Example ① $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$. $\quad f'(x) = \frac{1}{x\ln(b)}$

② $f(x) = x\ln(x) \quad f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

Hyperbolic trig functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

note: $\cosh^2 x - \sinh^2 x = 1$

$$\frac{d}{dx}\left(\sinh(x)\right) = \cosh(x) \qquad \frac{d}{dx}\left(\sinh(x)\right) = \cosh(x)$$