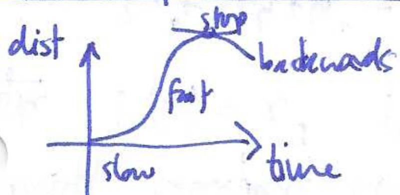
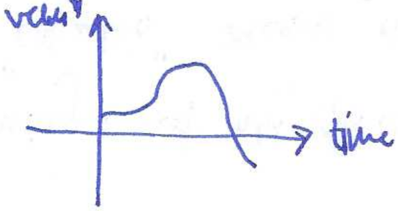
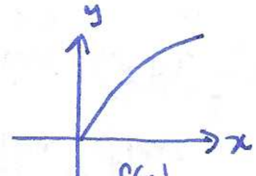
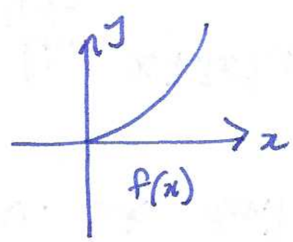
On the interpretation of graphs





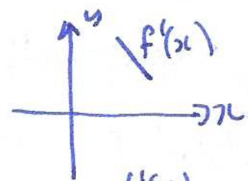


$f(x)$

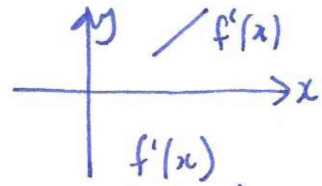


$f(x)$

← both increasing functions

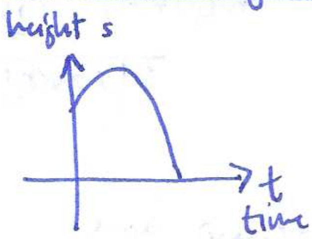


$f'(x)$ decreasing



$f'(x)$ increasing

Motion under gravity



$s_0 = s(0) =$ height at time $t=0$

$v_0 = v(0) =$ velocity at $t=0$

$\quad\quad\;\; \| $

$\quad\quad s'(0)$

$$s''(t) = v'(t) = a(t) = -g \qquad g = 9.8\ m/s^2,\ 32\ ft/s^2 \ \text{(constant)}$$

$$s'(t) = v(t) = -gt + v_0$$

$$s(t) = -\tfrac{1}{2}gt^2 + v_0 t + s_0$$

Example throw a stone upwards at 10 m/s from height 2m, what is max height?

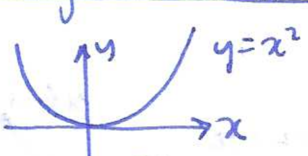
$s(t) = 2 + 10t - \frac{1}{2}gt^2$ (approx $g = 10$)

$v(t) = 10 - gt \qquad v(t) = 0 \quad t = \frac{10}{g} \sim 1 \qquad s(1) = 2+10-5 = 7m$

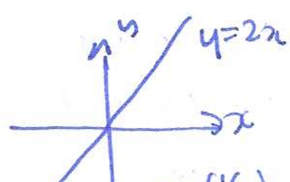
Q: how fast is it going when it hits the ground?

Q: If I can throw a stone 10m high, how fast can I throw it?

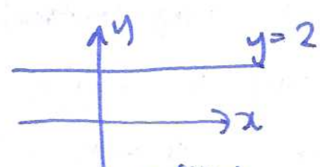
§3.5 Higher derivatives



"distance"



"velocity" slope of f



"acceleration". slope of f'

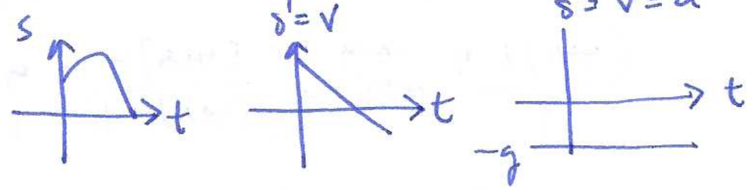
Example $f(x) = xe^x$

$$f'(x) = xe^x + e^x$$

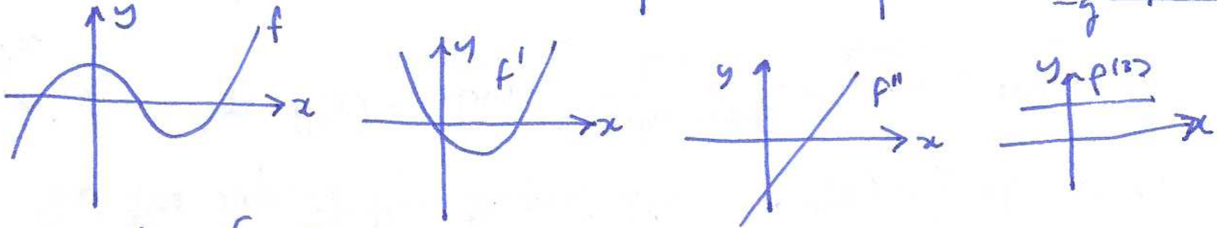
$$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$$

$$f^{(3)}(x) = xe^x + e^x + 2e^x = xe^x + 3e^x \quad \text{etc.}$$

Example acceleration due to gravity



Example



§3.6 Trigonometric functions

Thm $\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

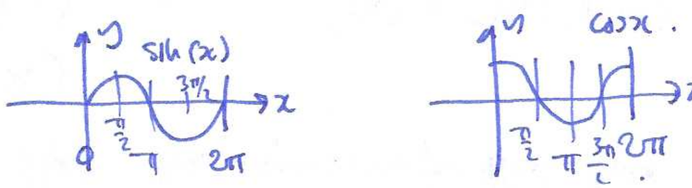
use: $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Proof (for $\sin(x)$) $\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} = \lim_{h\to 0} \sin x . \underbrace{\frac{\cos(h) - 1}{h}}_{0} + \lim_{h\to 0} \cos x \underbrace{\frac{\sin h}{h}}_{1}$$

$= \cos x$. $\square$.

Q: can this be right?



Example $f(x) = x\sin x$

$$f'(x) = x\cos x + \sin x.$$

Thm $\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Proof (of $\tan x$) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \ \square$

§3.7 Chain rule

composition of functions $f(g(x)) = f\circ g(x)$

Examples e^{4x}, $\sin^2 x$, etc. ...

Thm (chain rule) If f, g differentiable functions, then $f\circ g$ is differentiable and

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$ mnemonic: $(f(g(x)))' =$ outside$'$(inside) inside$'$.

note $f(g(x))$

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$x \mapsto g(x) \mapsto f(g(x))$$

local stretch factor $\quad g'(x) \quad f'(g(x))$

Examples ① $e^{4x} = f(g(x))$ where $f(x) = e^x$ $\quad f'(x) = e^x$

$g(x) = 4x$ $\quad g'(x) = 4$

so $(e^{4x})' = f'(g(x)) \cdot g'(x) = e^{4x} \cdot 4$

② $\sin^2 x = (\sin x)^2 = f(g(x))$ where $f(x) = x^2$ $\quad f'(x) = 2x$

$g(x) = \sin x$ $\quad g'(x) = \cos x$

so $(\sin^2 x)' = 2\sin x\cos x$

③ $\sqrt{x^3+1}$ ④ $(x+1)^{100}$

Alternate notation $f(g(x)) \longleftrightarrow f(u)$, $u = g(x)$

$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$ $\quad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$ mnemonic: "cancelling fractions"

Examples $\cos(x^2)$ $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^2+1}}$.

Proof (of chain rule)

$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$ $\quad$ [answer should be $f'(g(x))\cdot g'(x)$]