recall : e is defined as the special number s.t. the slope of e^x at $x=0$ is equal to 1, therefore if $\boxed{f(x)=e^x \text{ then } f'(x)=e^x}$ $\frac{d}{dx}(e^x)=e^x$

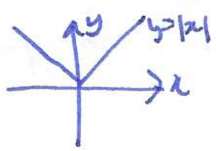
Example $\frac{d}{dx}(7e^x+8x^2) = 7e^x+16x$

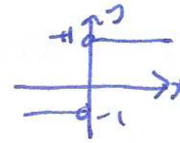
observation this shows that e^x is not a polynomial

$\frac{d^n}{dx^n}(p(x)) \leftarrow$ deg goes down by 1 each time, eventually get zero.

Thm Differentiable $\Rightarrow$ continuous $\square$ Warning : continuous $\not\Rightarrow$ differentiable.

example $f(x)=|x|$ cts





$f'(x)$ not cts at $x=0$

claim $f(x)=|x|$ not differentiable at $x=0$

check $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$ $\lim_{h\to 0^+}\frac{|h|}{h}=+1$

$\lim_{h\to 0^-}\frac{|h|}{h}=-1$ so $\lim_{h\to 0}\frac{|h|}{h}$ DNE. $\square$

local picture if $f(x)$ is differentiable at $x=c$, means $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists. (want to show $\lim_{x\to c}f(x)=f(c)$) consider $f(c+h)-f(c) = h\cdot\frac{f(c+h)-f(c)}{h}$

so $\lim_{h\to 0} f(c+h)-f(c) = \lim_{h\to 0} h\cdot\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} h \cdot \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = 0\cdot f'(c)=0$. $\square$.

§3.3. Product and quotient rules

new functions from old : $f(x)g(x) \leftarrow$ product $\frac{f(x)}{g(x)} \leftarrow$ quotient

Thm (product rule) $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$

$$(fg)' = f'g+fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

warning : $(fg)' \neq f'g'$!!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}\left((x)\cancel{\tfrac{d}{dx}}(x)\right) = \frac{d}{dx}(x).x + x\frac{d}{dx}(x) = 1.x+x.1 = 2x$.

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + (3x^2)(x^2+1)' = 6x(x^2+1) + 3x^2(2x)$.

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

<u>Proof</u> (of product rule) (assume f, g differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}g(x)$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x) = f(x)g'(x) + f'(x)g(x) \quad \square$$

<u>Thm</u> Quotient rule (assume f, g differentiable and $g(x) \neq 0$)

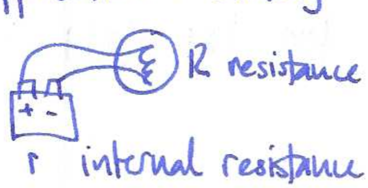
$$\left(\frac{f}{g}\right)'(x) = \frac{gf' - fg'}{g^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \quad \square$$

<u>Examples</u> ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x)(x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

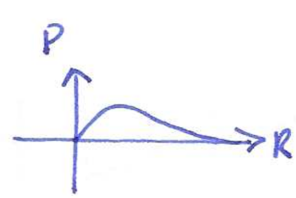
② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t)(e^t+t)'}{(e^t+t)^2} = \frac{(e^t+t)e^t - e^t(e^t+1)}{(e^t+t)^2}$

$$= \frac{te^t - e^t}{(e^t+t)^2}$$

③ application: battery power



power $P = \frac{V^2R}{(r+R)^2}$



Q: when does the battery give maximal power?

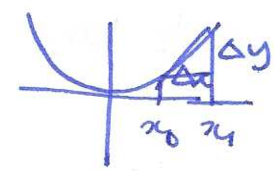
<u>A</u>: when $\frac{dP}{dR} = 0$ $\quad P = \frac{V^2R}{(r+R)^2}$ $\quad P(R)$, V, r constant

$$\frac{dP}{dR} = \frac{(r+R)^2 (v^2 R)' - ((r+R)^2)' \cdot (v^2 R)}{(r+R)^4} = \frac{(r+R)^2 (v^2) - (r^2+2rR+R^2)' (v^2 R)}{(r+R)^4}$$

$$= \frac{v^2}{(r+R)^4}\left(r^2+2rR+R^2 - (2r+2R)R \right) = \frac{v^2}{(r+R)^4}(r^2-R^2) = v^2 \frac{r-R}{(r+R)^4}$$

$\frac{dP}{dR} = 0$ when $r = R$ $\square$.

§3.4 Rates of change



recall: average rate of change $\frac{\Delta y}{\Delta x} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$

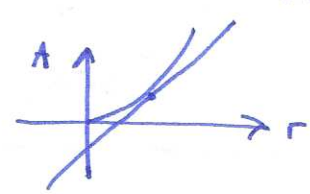
(instantaneous) rate of change: $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$

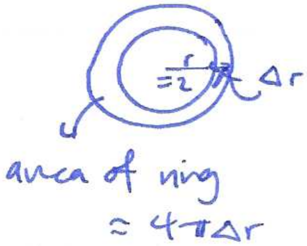
observation: if Δx is small, we can use the average rate of change to approximate the actual rate of change, and vice versa.

Example area of circle is πr^2: $A = \pi r^2$

calculate the rate of change of area wrt radius $\frac{dA}{dr} = 2\pi r$

e.g. $\left.\frac{dA}{dr}\right|_{r=2} = 4\pi$ $\quad \left.\frac{dA}{dr}\right|_{r=5} = 10\pi$





for small $\Delta r, h$: $f'(x_0) \approx \frac{f(x_0+h)-f(x)}{h}$

so $\boxed{f(x+h) \approx f(x_0) + h f'(x_0)}$

↑ linear approximation formula

Example stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$ (s speed in mph)

calculate stopping distance when $s = 30$ $\quad f(30) = 78$

$F'(s) = 1.1 + 0.1s$ $\quad F'(30) = 1.1 + 0.1 \times 30 = 4.1$ ft/mph

$F(s+h) \approx F(s) + hF'(s)$

$F(31) \approx F(30) + 1 \cdot F'(30) = 78 + 4.1 = 82.1$