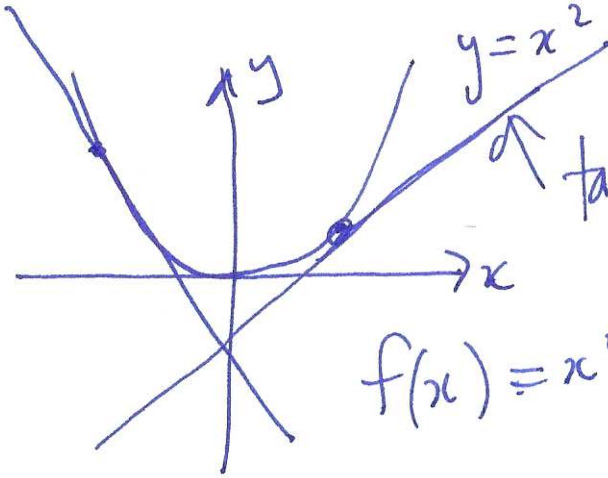
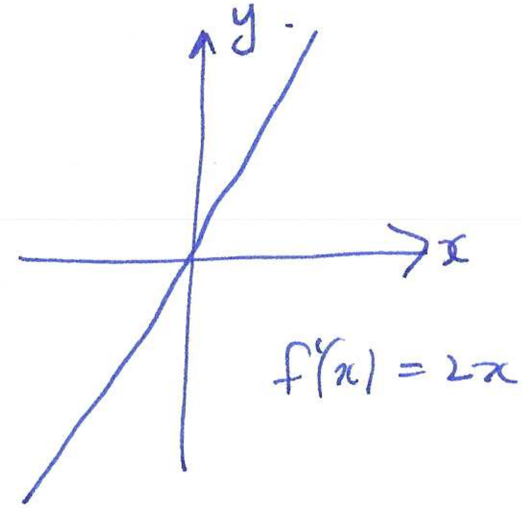
①

last time:





$x \longmapsto f(x)$

$f'(x) : x \longmapsto f'(x) \leftarrow$ slope of the tangent line

<u>recall</u>

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \qquad \frac{d}{dx}(kf) = k\frac{d}{dx}(f)$$

Exponential functions e^x b^x $b>0$

summary $\frac{d}{dx}(e^x) = e^x$ $e = 2.71\ldots$

consider $f(x) = b^x$ $b>0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h\to 0} \frac{b^x \cdot b^h - b^x}{h} = \lim_{h\to 0} \underbrace{b^x}\,\underbrace{\frac{b^h - 1}{h}}$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}$$

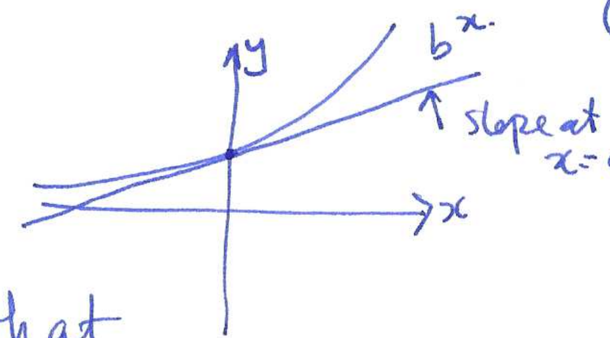
→ doesn't depend on x, only on b.

assume limit exists, call it μ_b

$$= \mu_b \, b^x$$

so: $\frac{d}{dx}(b^x) = \mu_b \, b^x$

$\uparrow$ constant.



recall e is the special number such that the slope of e^x at $x=0$ is equal to 1.

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$\mu_e = 1$

Example $\frac{d}{dx}(7e^x + 4x^3) = \frac{d}{dx}(7e^x) + \frac{d}{dx}(4\underbrace{x^3}_{3x^2})$.

$= 7e^x + 4 \times 3x^2 = 7e^x + 12x^2$

<u>observation</u> e^x is not a polynomial

$$\frac{d}{dx}\left(p(x)\right) \leftarrow \text{deg } n-1$$

$\uparrow$ deg n.

degree goes down by 1 when you differentiate.

<u>Example.</u>

$f(x) = x^3$.

$f'(x) = 3x^2$

$f''(x) = 6x$

$f^{(3)}(x) = 6$

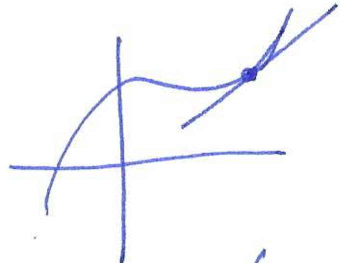
$f^{(4)}(x) = 0$

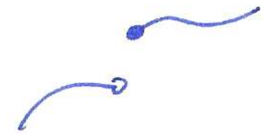
all polynomials eventually differentiate to zero.

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d^k}{dx^k}(e^x) = e^x \leftarrow \underline{\text{not}}\ \text{zero function.}$$

useful fact (Thm) $f(x)$ Differentiable $\Rightarrow$ $f(x)$ continuous $\square$.

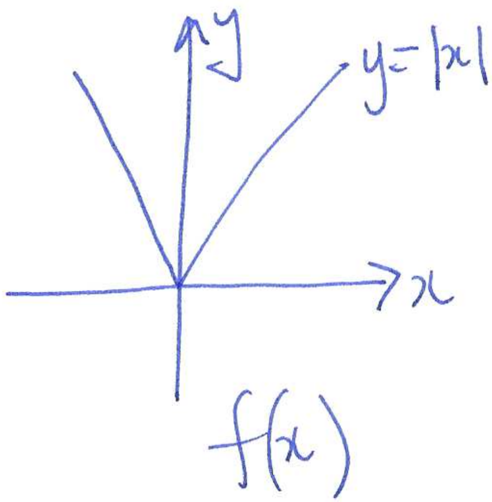


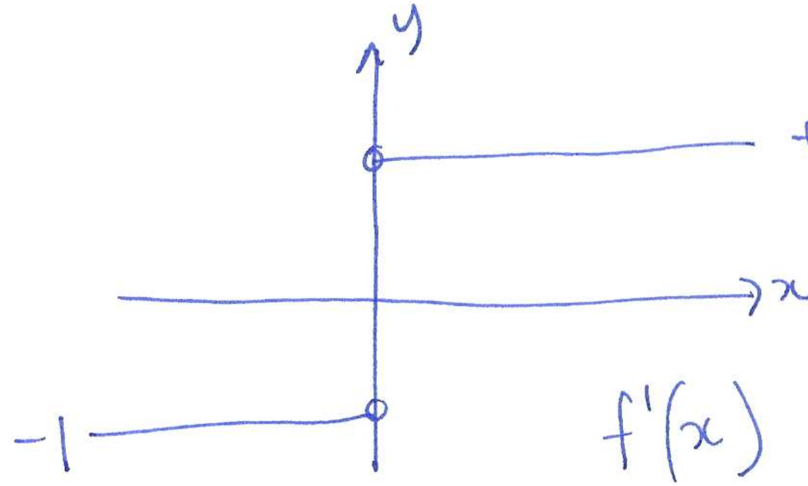


warning continuous $\not\Rightarrow$ differentiable

Example

$f(x) = |x|$





$f'(0)$ DNE.

§3.3 Product and quotient rules

Let $f(x)$ and $g(x)$ be differentiable functions.

$f(x)\,g(x)$ product $\qquad \dfrac{f(x)}{g(x)}$ quotient

Thm (product rule)

$$(f(x)g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

warning

$(fg)' \neq f' \cdot g'$ ⍨

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x)$ $\boxed{(fg)' = f'g + fg'}$

product rule

$$= \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = 1 \cdot x + x \cdot 1 = 2x.$$

② $$\frac{d}{dx}\Big(\underbrace{3x^2}_{f}\underbrace{(x^2+1)}_{g}\Big) = \underbrace{6x(x^2+1)}_{f'g} + \underbrace{3x^2(2x)}_{fg'}.$$

$$= 6x(x^2+1) + 6x^3$$

③ $$\frac{d}{dx}\Big(\underbrace{x^2}_{f}\underbrace{e^x}_{g}\Big)$$

$$= \boxed{\underbrace{2x\, e^x}_{f'g} + \underbrace{x^2\, e^x}_{fg'}}$$

product rule

use:

$$\frac{d}{dx}\left(\underbrace{x^2}_{f}\underbrace{e^x}_{g}\right). \qquad (fg)' = f'g + fg'.$$

$$= \overset{f'}{\underbrace{\frac{d}{dx}(x^2)}_{2x\cancel{x}}} \cdot \overset{g}{e^x} + \overset{f}{x^2}\,\overset{g'}{\frac{d}{dx}(e^x)}.$$

$$\ 2x \cdot e^x \qquad x^2 e^x$$

using power of x

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$= 2xe^x + x^2e^x.$$

Proof (of the product rule) f, g differentiable functions.

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \underbrace{f(x+h)}_{f(x)} \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)} + \underbrace{\frac{f(x+h) - f(x)}{h}}_{f'(x)} \underbrace{g(x)}_{g(x)}.$$

$$= fg' + f'g \quad \square.$$

Thm (quotient rule) assume f, g differentiable

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}}$$

product rule.

assuming $g(x) \neq 0$.

Examples (1) $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \left(\frac{f(x)}{g(x)}\right)'$ $\quad f(x) = x$, $g(x) = x+1$.

$$\frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2} = \frac{(x+1) \cdot (x)' - x(x+1)'}{(x+1)^2}.$$

$$= \frac{(x+1) \cdot (1) - (x)(1)}{(x+1)^2}$$

$$= \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2} = \boxed{\frac{1}{(x+1)^2}}$$

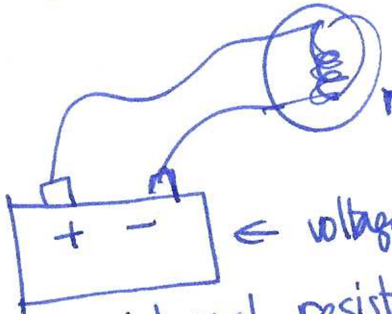
2 $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right)$ $\quad \left(\frac{f(t)}{g(t)}\right)' = \frac{gf' - fg'}{g^2}$

$$= \frac{\overset{g}{(e^t+t)}\overset{f'}{(e^t)'} - \overset{f}{(e^t)}\overset{g'}{(e^t+t)'}}{\underset{g^2}{(e^t+t)^2}}$$

$$= \frac{(e^t+t)(e^t) - e^t(e^t+1)}{(e^t+t)^2}$$

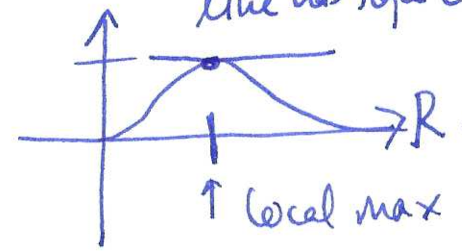
$$= \frac{\cancel{e^{2t}} + te^t - \cancel{e^{2t}} - e^t}{(e^t+t)^2} = \boxed{\frac{e^t(t-1)}{(e^t+t)^2}}$$

③ applications : battery power.



power $P = \dfrac{V^2 R}{(r+R)^2}$. $\quad \dfrac{f}{g}$

$P(R)$



assume V, r fixed but can vary R.

find

$$\frac{dP}{dR} = \frac{(r+R)^2 (V^2R)' - V^2R\left((r+R)^2\right)'}{\left((r+R)^2\right)^2}.$$

use $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$.

$f = V^2 R$.

$g = (r+R)^2$

$$= \frac{(r+R)^2 V^2 - V^2R\left(r^2 + 2rR + R^2\right)'}{(r+R)^4}$$

$$= \frac{(r+R)^2 V^2 - V^2R(2r+2R)}{(r+R)^4}.$$

$$= \frac{V^2\left[r^2 + \cancel{2rR} + R^2 - \cancel{2rR} - 2R^2\right]}{(r+R)^4}.$$

$(r-R)(r+R)$

$$= \frac{V^2(r^2 - R^2)}{(r+R)^4} = \boxed{V^2 \frac{r-R}{(r+R)^3}}$$

$$\frac{dP}{dR} = V^2 \frac{r-R}{(r+R)^3} = 0 \quad \text{when} \quad r = R.$$

<u>summary</u> battery gives max power when $r = R$.

<u>use</u>. $(fg)' = f'g + fg'$.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

④ <u>Example</u>. $h(x) = \left(\dfrac{\overbrace{xe^x + \dfrac{x^2}{e^{x}+1}}^{f}}{\underbrace{x^2+2}_{g}} \right)'.$

$$= \frac{(x^2+2)\left[xe^x + \dfrac{x^2}{e^x+1}\right]' - \left[xe^x + \dfrac{x^2}{e^x+1}\right](x^2+2)'}{(x^2+2)^2}.$$

$(x^2+2)' = 2x.$

$$\left[xe^x + \frac{x^2}{e^x+1}\right]' = (x)(e^x)' + (x)'(e^x) + \frac{(e^x+1)(x^2)' - x^2(e^x+1)'}{(e^x+1)^2}.$$

$$= e^x x + e^x + \frac{(e^x+1)(2x) - x^2 e^x}{(e^x+1)^2}$$

$$h'(x) = \frac{(x^2+2)\left[e^x(x+1) + \frac{(e^x+1)2x - x^2 e^x}{(e^x+1)^2}\right] - \left(xe^x + \frac{x^2}{e^x+1}\right)2x}{(x^2+2)^2}$$

Summary. $(x^n)' = nx^{n-1}$ $\quad (e^x)' = e^x.$

$(f+g)' = f' + g'$ $\quad (kf)' = kf'$ $\quad (fg)' = f'g + fg'.$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

other useful rules. $\frac{d}{dx}(\sin(x)) = \cos(x)$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

chain rule.

composition of functions $(f(g(x)))' = f'(g(x)) \cdot g'(x)$.

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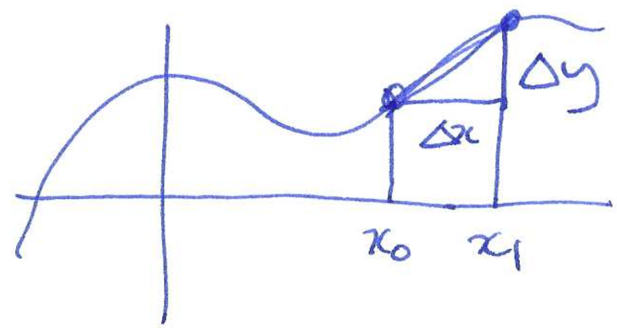
§3.4 Rates of change

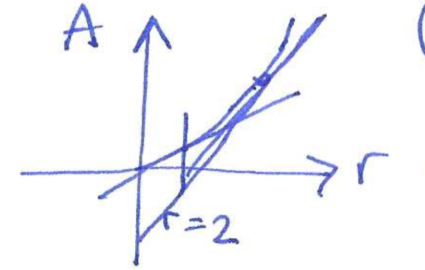


recall : average rate change.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

rate of change $\lim_{\substack{x_1 \to x_0 \\ \Delta x \to 0}} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$.

observation if Δx is small, can use average rate of change to approx. the rate of change



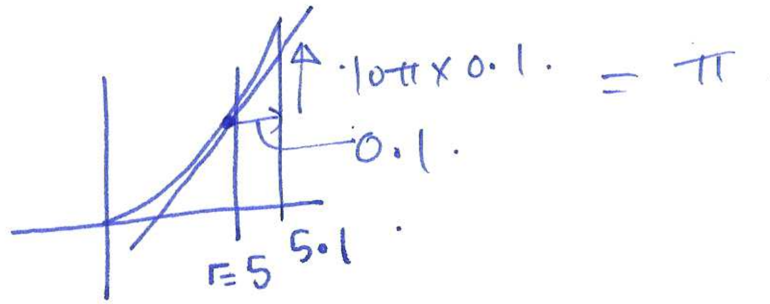
Example: area of circle $A = \pi r^2$

we can calculate the rate of change of area with respect to radius

$$\frac{dA}{dr} = 2\pi r \quad \text{e.g.} \quad \left.\frac{dA}{dr}\right|_{r=2} = 4\pi \qquad \left.\frac{dA}{dr}\right|_{r=5} = 10\pi.$$

at $r=5$ $\quad A = 25\pi$.

what about at $r=5.1$?



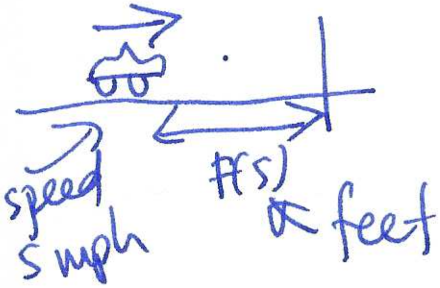
$$A(5.1) \approx 25\pi + \pi.$$

in general for small h: $\quad f'(x_0) \approx \dfrac{f(x_0+h) - f(x)}{h}$.

linear approx formula

$$\boxed{f(x_0+h) \approx f(x_0) + h f'(x_0)}$$

Example stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$

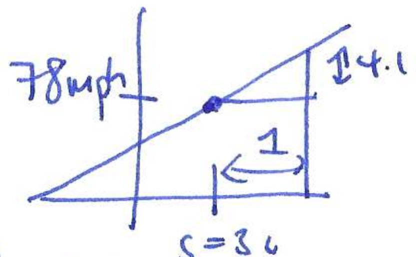


stopping distance at 30 mph is $F(30)$

$= 1.1 \times 30 + 0.05 \times 30^2 = 78$ ft.

$F'(s) = 1.1 + 0.1s$ $\qquad F'(30) = 1.1 + 0.1 \times 30 = 4.1$ ft/mph.

$F(s+h) \approx F(s) + hF'(s)$



$F(31) \approx F(30) + 1 \cdot F'(30) = 78 + 4.1$

$= 82.1$ ft.