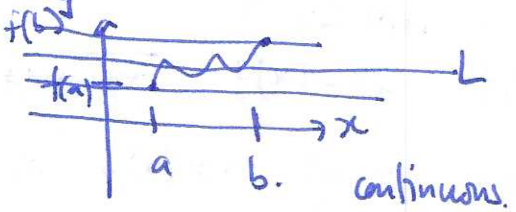
(2) $\lim_{x\to\infty} \frac{x^2+x}{x-3} = \lim_{x\to\infty} \frac{x+1}{1-3/x} = +\infty$

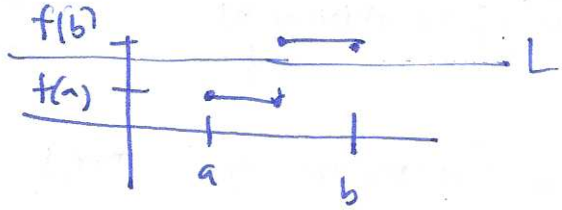
(3) $\lim_{x\to\infty} \frac{1}{x} - \frac{2}{3x+1} = \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{2}{3x+1} = 0-0=0$

(4) $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{4x+1} = \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{4+1/x} = \frac{\sqrt{2}}{4}$ (do $\lim_{x\to-\infty}$!).

§2.8 Intermediate value theorem (IVT)

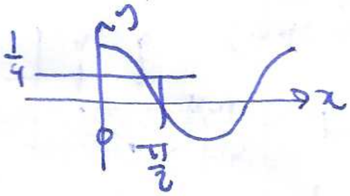
"continuous functions can't skip values"





Thm (IVT) If $f(x)$ is a cts function on a closed interval $[a,b]$ with $f(a)\neq f(b)$ then for any number L, with $f(a) \le L \le f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = L$ $\square$.

Example show $\cos(x) = \frac{1}{4}$ has at least one solution (in $[0,\frac{\pi}{2}]$)



$\cos(0) = 1$
$\cos(\frac{\pi}{2}) = 0$

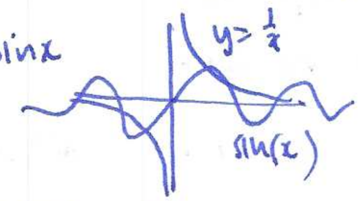
$0 \le \frac{1}{4} \le 1$ IVT $\Rightarrow$ there is a $c \in [0,\frac{\pi}{2}]$ s.t. $f(c) = \frac{1}{4}$.

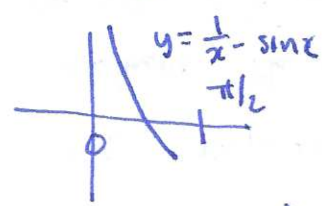
special case : finding zeros

Corollary if $f(x)$ is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have different signs, then there is at least one $c \in [a,b]$ s.t. $f(c) = 0$ $\square$.

Bisection method find a solution to $\sin x = \frac{1}{x}$ in $[0,\frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin x$



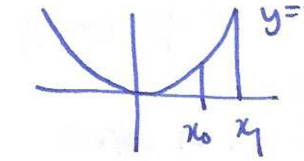


$f(0) = +\infty > 0$ $\quad f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 < 0$. Midpoint : $f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.5665 > 0$

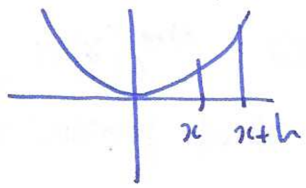
so at least one solution in $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc....

§3.1 Def^{n} of the derivative

Recall: we can compute the average rate of change of a function over an interval



$[x_0, x_1]$ $\qquad \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$



$[x, x+h]$ $\qquad \dfrac{f(x+h)-f(x)}{h}$

Q: how do we find the slope of the tangent line?

A: look at the average rate of change over a small interval $[x, x+h]$ and take the limit as $h \to 0$

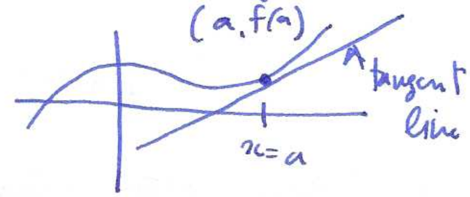
Def^{n} the slope of the tangent line at $x=a$ is $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

notation: also called the *derivative* at a, written $f'(a)$ (Newton) or $\dfrac{df(a)}{dx}$ (Leibniz).

If this limit exists we say the function is differentiable at $x=a$.

Note: $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Def^{n} the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$



The equation for this line is:

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

Example find the tangent line to $y=x^2$ at $x=1$

$(x, f(x)) = (1,1)$ $\quad$ slope: $\displaystyle f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1}{h}$

$\displaystyle = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2+h = 2.$