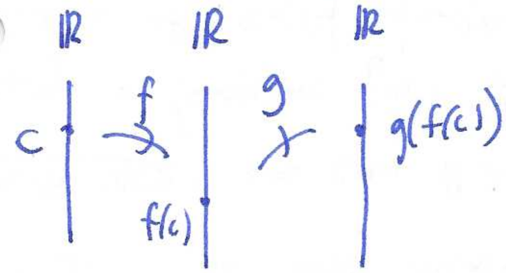
Thm 5 (composition) If $f(x)$ is continuous at $x=c$, and $g(x)$ is continuous at $x=f(c)$, then $g(f(x))$ is continuous at $x=c$.




Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x+1}}$ cts at $x=1$

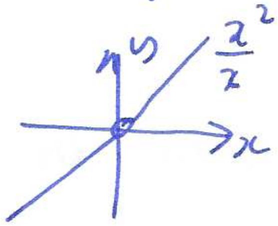
Q: where is $f(x) = \dfrac{x^2}{\sin(x)}$ cts?

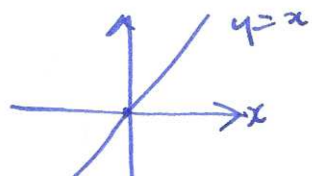
## §2.5 Evaluating limits algebraically

Example $\dfrac{x^2}{x}$ undefined at $x=0$ $\quad \dfrac{0}{0} \leftarrow$ indeterminate form

but $\lim_{x \to 0} \dfrac{x^2}{x}$ does not depend on value at $x=0$

$\dfrac{x^2}{x} = x$ for $x \neq 0$




so $\lim_{x \to 0} \dfrac{x^2}{x} = \lim_{x \to 0} x = 0$

indeterminate forms : $\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 0^0$

note : $\dfrac{1}{0}$ not indeterminate form, limit will be $\pm\infty$ or DNE.

Examples ① $\lim_{x \to 3} \dfrac{x^2 - 4x + 3}{x^2 + x - 12}$ $\quad x=3: \dfrac{9-12+3}{9+3-12} = \dfrac{0}{0}$

factor: $\dfrac{(x-3)(x-1)}{(x-3)(x+4)} = \dfrac{x-1}{x+4}$ $\quad (x \neq 3)$

$\lim_{x \to 3} \dfrac{(x-3)(x-1)}{(x-3)(x+4)} = \lim_{x \to 3} \dfrac{x-1}{x+4} = \dfrac{2}{7}$

② $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$ $\quad x=4: \dfrac{2-2}{4-4} = \dfrac{0}{0}$

$$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} \quad (x \neq 4)$$

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

③ $$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x/\cos x}{1/\cos x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$$

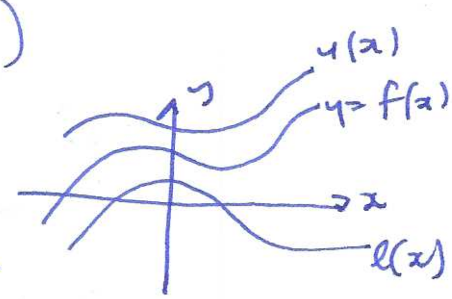
④ $$\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1} \quad (x \neq 1)$$

$$\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$$

## §2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin x}{x}$ ? $x=0$ get $\frac{0}{0}$ indeterminate form.

A $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ( try $\pm 0.1$, $\pm 0.01$, $\pm 0.001$, ... )

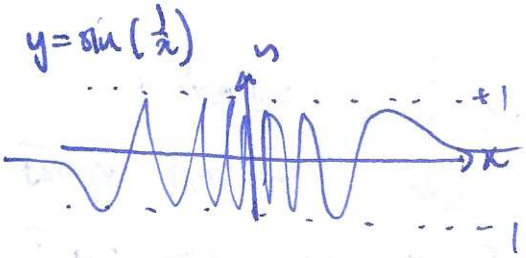


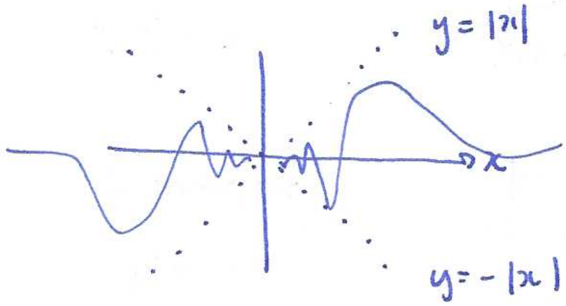

**squeeze Thm** suppose $l(x) \leq f(x) \leq u(x)$

and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$. Then $\lim_{x\to c} f(x) = L$

**Examples** $\lim_{x\to 0} x\sin\frac{1}{x}$

$y = \sin\left(\frac{1}{x}\right)$



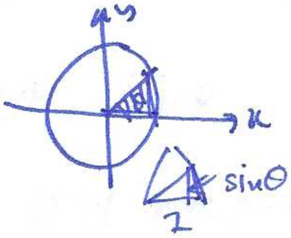



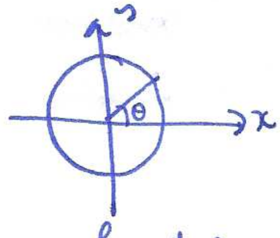

$-1 \leq \sin x \leq +1$ $\quad \lim_{x\to 0} |x| = 0 = \lim_{x\to 0} -|x| \Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

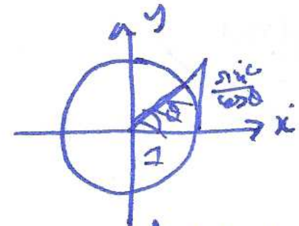
$-|x| \leq x\sin(x) \leq |x|$

**Thm** $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$

**Proof** (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$, consider the following three areas:




area of triangle $\frac{1}{2}bh$ $\qquad \frac{1}{2}\cdot 1 \sin\theta \quad \leq \quad$ area of sector $\pi r^2 \cdot \frac{\theta}{2\pi} \quad \leq \quad$ area of triangle $\frac{1}{2}bh$ $\qquad \frac{1}{2}\frac{\sin\theta}{\cos\theta}$.

regions are subsets of each other:

$$\underbrace{\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta}_{\frac{\sin\theta}{\theta} \leq 1} \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\cos\theta \leq \frac{\sin\theta}{\theta}$$

so $\quad \cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$ $\qquad$ squeeze thm $\Rightarrow \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$\lim\limits_{\theta\to 1} \cos\theta = 1 \qquad \lim\limits_{\theta\to 1} 1 = 1$

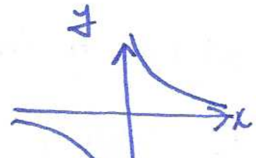
<u>Examples</u> ① $\lim\limits_{x\to 0} \frac{\sin 3x}{2x}$ $\qquad$ know: $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

write: $3x = \theta \qquad \lim\limits_{\theta\to 0} \frac{\sin\theta}{2\cdot\theta/3} = \lim\limits_{\theta\to\infty} \frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}$

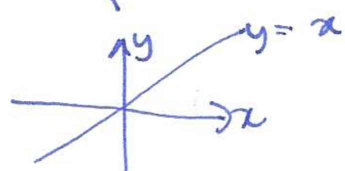
② $\lim\limits_{t\to 0} \frac{1-\cos t}{\sin t} = \lim\limits_{t\to 0} \frac{1-\cos t}{t}\cdot\frac{t}{\sin t} = \underbrace{\lim\limits_{t\to 0}\frac{1-\cos t}{t}}_{0} \cdot \frac{1}{\underbrace{\lim\limits_{t\to 0}\frac{\sin t}{t}}_{1}} = 0$

<u>§2.7 Limits at infinity</u>

key observations: $\lim\limits_{x\to 0} \frac{1}{x} = 0$




$\lim\limits_{x\to\infty} x = +\infty$




<u>Examples</u> ① $\lim\limits_{x\to\infty} \frac{3x}{2x-1} = \lim\limits_{x\to\infty} \frac{3}{2-1/x} = \frac{3}{2-\lim\limits_{x\to\infty}\frac{1}{x}} = \frac{3}{2}$