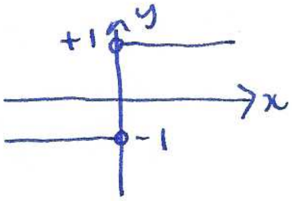
One sided limits example $f(x) = \frac{x}{|x|}$ $f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$



sometimes useful to distinguish left/right/two sided limits

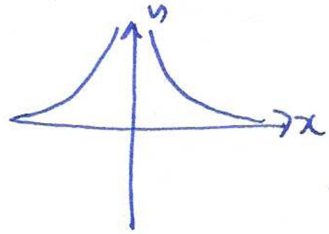
notation $\lim\limits_{x \to 0+} f(x)$ means right limit $(x>0)$

$\lim\limits_{x \to 0-} f(x)$ means left limit $(x<0)$

note: the two sided limit exists if and only if the left and right limits exist and are equal.

example $f(x) = \frac{x}{|x|}$ $\left.\begin{array}{l} \lim\limits_{x\to 0+} f(x) = +1 \\ \lim\limits_{x \to 0-} f(x) = -1 \end{array}\right\}$ $1 \neq -1$ so $\lim\limits_{x \to 0} f(x)$ DNE.

example $f(x) = \frac{1}{x^2}$



$\left.\begin{array}{l} \lim\limits_{x \to 0+} \frac{1}{x^2} = +\infty \\ \lim\limits_{x \to 0-} \frac{1}{x^2} = +\infty \end{array}\right\}$ $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$.

§2.3 Basic limit laws

Example $\lim\limits_{x \to 0} 2x+3 = \lim\limits_{x\to 0} 2x + \lim\limits_{x \to 0} 3 = 2\lim\limits_{x \to 0} x + \lim\limits_{x \to 0} 3 = 2\cdot 0 + 3 = 3$

Thm assume that $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist and are finite. Then:

1) sums: $\lim\limits_{x \to c} f(x) + g(x) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

2) constant multiple: $\lim\limits_{x \to c} k f(x) = k \lim\limits_{x \to c} f(x)$ k constant (does not depend on x)

3) products: $\lim\limits_{x \to c} (f(x) g(x)) = (\lim\limits_{x \to c} f(x))(\lim\limits_{x \to c} g(x))$.

4) quotients: $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ as long as $\lim\limits_{x \to c} g(x) \neq 0$

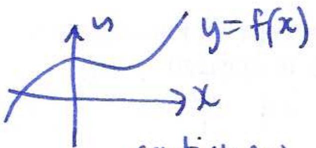
warning: these rules don't work if either $\lim\limits_{x \to c} f(x)$ or $\lim\limits_{x \to c} g(x)$ DNE.

Examples $\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 3.3 = 9$

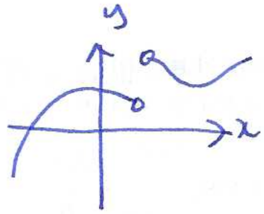
$$\lim_{t\to 2}\frac{t+5}{2t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 2t} = \frac{\lim_{t\to 2} t + 5}{2\lim_{t\to 2} t} = \frac{2+5}{2\times 2} = \frac{7}{4}$$

§2.4 Limits and continuity

Example



continuous

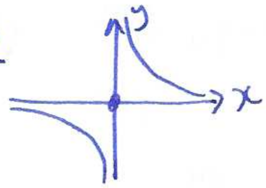


not continuous | discontinuous.

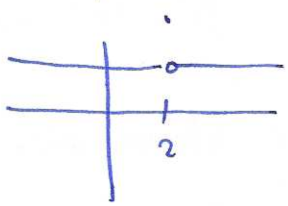
Defn We say $f(x)$ is continuous at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

If the limit is DNE, infinite, or not equal to $f(c)$, then $f(x)$ is not continuous at $x=c$

Examples



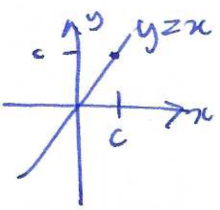
$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & \text{at } x = 0 \end{cases}$ $\leftarrow$ not cts at $x=0$



$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2 \end{cases}$ $\lim_{x\to 2} f(x) = 1 \neq f(2)$

not cts at $x = 2$.

Example show $f(x) = x$ is continuous



$f(c) = c$, need $\lim_{x\to c} f(x) = c$

follows from limit laws: $\lim_{x\to c} x = c$ ✓.

Corollary: polynomials | rational functions are continuous where defined.

Defn $f(x)$ is left continuous at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$

right continuous at $x=c$ if $\lim_{x\to c^+} f(x) = f(c)$

If at least one of the left/right limits is $\pm\infty$, we say $f(x)$ has an infinite discontinuity at $x=c$.

Building continuous functions

Thm 0 $f(x)=k$, $f(x)=x$ are continuous.

Thm 1 suppose $f(x), g(x)$ are continuous at $x=c$: The following functions are cts at c :

1) $f(x)+g(x)$ 2) $kf(x)$ k constant

3) $f(x)\,g(x)$ 4) $\dfrac{f(x)}{g(x)}$ if $g(c)\neq 0$

Proof these follow immediately from the limit laws

check 1) $f(x)$ continuous means $\lim_{x\to c} f(x) = f(c)$

$g(x)$ " $\lim_{x\to c} g(x) = g(c)$

so $\lim_{x\to c} (f(x)+g(x)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right) = f(c)+g(c)$, as required □

Thm 2 Polynomials are continuous. $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

Rational functions $\dfrac{p(x)}{q(x)}$ are continuous where $q(x)\neq 0$

Proof $f(x)=x$ is continuous. so $f(x).f(x) = x^2$ is cts. (products)

so $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is cts (products; mult by constants; sums)

so $\dfrac{p(x)}{q(x)}$ is continuous (quotients) where $q(x)\neq 0$ □.

Useful facts

Thm 3
- $\sin(x), \cos(x)$ are continuous
- $x^{1/n}$ is continuous
- b^x is continuous
- $\log_b(x)$ is cts.

combinations of these with polynomials sometimes called *elementary functions*

Thm 4 (Inverse functions) If $f: \mathcal{D} \to R$ is continuous, with inverse $f^{-1}: R \to \mathcal{D}$, then f^{-1} is continuous

$\mathcal{D} \to R \subseteq \mathbb{R}$, $\subseteq \mathbb{R}$