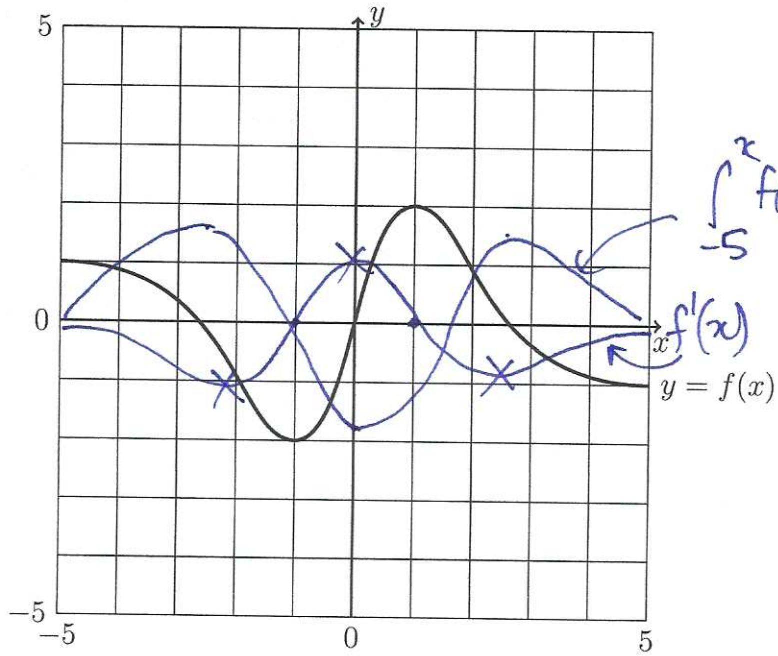
Math 231 Calculus 1 Spring 26 Sample Midterm 3

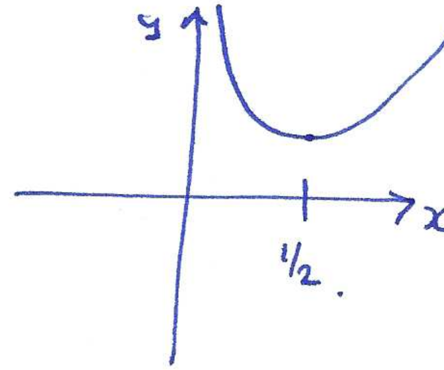
(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-5,-1) \cup (1,5)$.
(b) Label all regions where $f'(x) > 0$. $(-1,1)$.
(c) What is $\lim_{x\to\infty} f(x)$? -1
(d) What is $\lim_{x\to-\infty} f'(x)$? 0
(e) What is $\lim_{x\to\infty} f''(x)$? 0
(f) Sketch a graph of $f'(x)$ on the figure.
(g) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.
(h) Label the approximate locations of all points of inflection. (×) on graph

Q2 $f(x) = 2x^2 - \ln(x)$.

a) vertical asymptote at $x=0$
no horizontal asymptote.



b) $f'(x) = 4x - \frac{1}{x}$
critical pt: solve $f'(x)=0$: $4x - \frac{1}{x} = 0$
$x^2 = \frac{1}{4}$ $x = \frac{1}{2}$

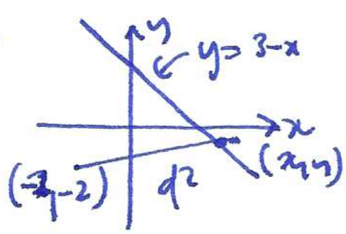
c) $f'(x) > 0$ $(\frac{1}{2}, \infty)$
$f'(x) < 0$ $(0, -\frac{1}{2})$

d) $f''(x) = 4 + \frac{1}{x^2}$ inflection pt: $f''(x)=0$ no solution, no inflection pts.

e) $f''(x) > 0$ concave up.

f) local min at $x = \frac{1}{2}$

Q3



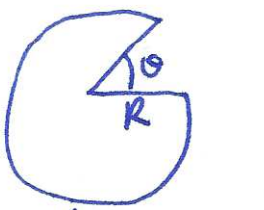
$d^2 = (x-(-1))^2 + (y-(-2))^2 = (x+1)^2 + (y+2)^2$

$d^2 = (x+1)^2 + (3-x+2)^2 = (x+1)^2 + (5-x)^2$

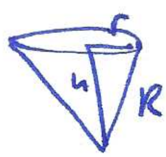
$\frac{d}{dx}(d^2) = 2(x+1) + 2(5-x).(-1)$

critical pt: $\frac{d}{dx}(d^2) = 0$: $2x + 2 - 10 + 2x = 0$ $4x = 8$ $x = 2$, $y = 1$.

Q4



$(2\pi - \theta)R$.



$2\pi r = (2\pi - \theta)R \Rightarrow r = (1 - \frac{\theta}{2\pi})R$.

$R^2 = h^2 + r^2$

$V = \frac{2}{3}\pi r^2 h$

$V = \frac{2}{3}\pi \left(1 - \frac{\theta}{2\pi}\right)^2 R^2 \sqrt{R^2 - \left(1 - \frac{\theta}{2\pi}\right)^2 R^2}$

set $\phi = 1 - \frac{\theta}{2\pi}$

$V = \frac{2}{3}\pi R^3 \phi^2 \sqrt{1 - \phi^2}$

$\frac{dV}{d\phi} = \frac{2}{3}\pi R^3 \left(2\phi\sqrt{1-\phi^2} + \phi^2 \frac{1}{2}(1-\phi^2)^{-1/2}.(-2\phi)\right)$

critical pt: $\frac{dV}{d\phi} = 0$: $2\phi\sqrt{1-\phi^2} = \frac{\phi^3}{\sqrt{1-\phi^2}}$. $\Rightarrow 2(1-\phi^2) = \phi^2$ $\phi^2 = \frac{2}{3}$
$\phi = \sqrt{2/3}$.

$$\theta = 2\pi\left(1-\sqrt{2/3}\right)$$

Q5 a) $$\lim_{x\to -3} \frac{2x^2+5x-3}{3x^2+10x+3} \overset{l'H}{=} \lim_{x\to -3} \frac{4x+5}{6x+10} = \frac{-7}{-8} = \frac{7}{8}$$

b) $$\overset{l'H}{=} \lim_{x\to 0} \frac{3}{\sec^2(4x)\cdot 4} = \frac{3}{4}.$$

c) $$\lim_{x\to 9} \frac{3-\sqrt{x}}{x-9} = \lim_{x\to 9} \frac{3-\sqrt{x}}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{-1}{\sqrt{x}+3} = \frac{-1}{6}.$$

d) $$\lim_{x\to 1} \frac{3(x-1)(2x+1)}{1-1/x} = \lim_{x\to 1} \frac{3x(x-1)(2x+1)}{(x-1)} = \lim_{x\to 1} 3x(2x+1) = 9$$

e) $$\overset{l'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+9x^2}\cdot 3}{\frac{1}{\sqrt{1-4x^2}}\cdot 2} = \frac{3}{2}$$

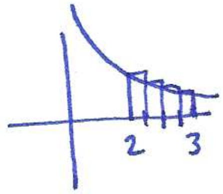
f) $$\lim_{x\to\infty} \frac{\ln(x)^2}{\sqrt{x}} \overset{l'H}{=} \lim_{x\to\infty} \frac{2\ln(x)\cdot\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{4\ln(x)}{x^{1/2}} \overset{l'H}{=} \lim_{x\to\infty} \frac{4/x}{\frac{1}{2}x^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{8}{\sqrt{x}} = 0$$

g) $$\lim_{x\to 2^-} \frac{\tan(\pi x)}{\ln(2-x)} \overset{l'H}{=} \lim_{x\to 2^-} \frac{\sec^2(\pi x)\cdot\pi}{\frac{1}{2-x}\cdot -1} = \lim_{x\to 2^-} \frac{-\pi(2-x)}{\cos^2(\pi x)} = 0$$

h) $$= \lim_{x\to\infty} \frac{3-2/x+1/x^2}{\sqrt{x+4/x-2/x^4}} = 0.$$

Q6

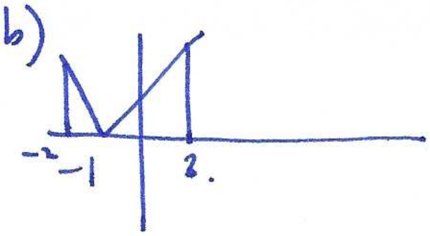


over estimate

$$\sim \frac{1}{4}\left(f(2)+f(2+\tfrac{1}{4})+f(2+\tfrac{2}{4})+f(2+\tfrac{3}{4})\right)$$

$$= \frac{1}{4}\left(\frac{1}{2}+\frac{4}{9}+\frac{2}{5}+\frac{4}{11}\right)$$

Q7 a) $$\int 2x^{8/3} + x^{2/3} - 3x^{-1/3}\,dx = 2\cdot\frac{3x^{11/3}}{11} + \frac{3}{5}x^{5/3} - 3\cdot\frac{3}{2}x^{2/3} + c$$

b) 

$$\int_{-2}^{-1} -(x+1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$\frac{1}{2} + 8 = 8\tfrac{1}{2}.$$

c) $$\int_1^4 2x^{-1/2}\,dx = \left[4x^{1/2}\right]_1^4 = 8-4 = 4.$$

d) $$\int_1^3 e^{-2x}\,dx = \left[-\frac{1}{2}e^{-2x}\right]_1^3 = -\frac{1}{2}e^{-6} + \frac{1}{2}e^{-2}$$

e) $$\int_0^x \frac{1}{t+2}\,dt = \Big[\ln|t+2|\Big]_0^x = \ln(x+2) - \ln(2).$$

f) $$\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{1}{1+(x/2)^2}\,dx \qquad u = \frac{x}{2} \quad \frac{du}{dx} = \frac{1}{2}.$$

$$= \frac{1}{4}\int \frac{1}{1+u^2}\frac{dx}{du}\,du = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan(u)+c = \frac{1}{2}\tan\left(\frac{x}{2}\right)+c$$

g) $$\int \frac{x}{1+9x^2}\,dx \qquad u = 1+9x^2 \quad \frac{du}{dx} = 18x \qquad \int \frac{x}{u}\frac{dx}{du}\,du = \int \frac{x}{u}\cdot\frac{1}{18x}\,du = \frac{1}{18}\int \frac{1}{u}\,du$$

$$= \frac{1}{18}\ln|u|+c$$

h) $$\int \cos(2x)\,dx \qquad u = 2x \quad \frac{du}{dx} = 2 \qquad \int \cos(u)\frac{dx}{du}\,du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u)+c$$

$$= \frac{1}{2}\sin(2x)+c$$

i) $$\int x\sin(1+x^2)\,dx \qquad u = 1+x^2 \quad \frac{du}{dx} = 2x \qquad \int x\sin(u)\frac{dx}{du}\,du = \int x\sin(u)\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u)+c = -\frac{1}{2}\cos(1+x^2)+c$$

j) $$\int \sin(x)\cdot e^{-\cos(x)}\,dx \qquad u = -\cos(x) \quad \frac{du}{dx} = +\sin(x) \qquad \int \sin(x)\,e^u\cdot\frac{dx}{du}\,du = \int \sin(x)\cdot e^u\cdot\frac{1}{\sin(x)}\,dx$$

$$= \int e^u \, du = e^u + c = e^{-\cos(x)} + c.$$

k) $\int \frac{\sin(x)}{\cos^4(x)} dx$ $\quad u = \cos(x)$, $\frac{du}{dx} = -\sin(x)$ $\quad \int \frac{\sin(x)}{u^4} \frac{dx}{du} du = \int \frac{\sin(x)}{u^4} \frac{1}{-\sin(x)} du$

$$= \int -u^{-4} du = \frac{1}{3} u^{-3} + c = \frac{1}{3\cos^3 x} + c$$

<u>Q8</u> $A(x) = \int_0^x e^{-2t} \cos(t) dt \qquad A'(x) = e^{-2x} \cos(x)$

$$(A(\sqrt{x}))' = A'(\sqrt{x}) \cdot \tfrac{1}{2} x^{-1/2} = e^{-2\sqrt{x}} \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

<u>Q9</u> $v(t) = \frac{1}{(t+2)^3}$.

$x(t) = \int \frac{1}{(t+2)^3} dt$ $\quad u = t+2$, $\frac{du}{dt} = 1$ $\quad = \int \frac{1}{u^3} du = -\frac{1}{2u^2} + c = -\frac{1}{2(t+2)^2} + c$

$x(0) = 0 \Rightarrow c = \frac{1}{8}$

$\lim_{x \to \infty} \frac{1}{8} - \frac{1}{2(t+2)^2} = \frac{1}{8}$, doesn't get to 10.