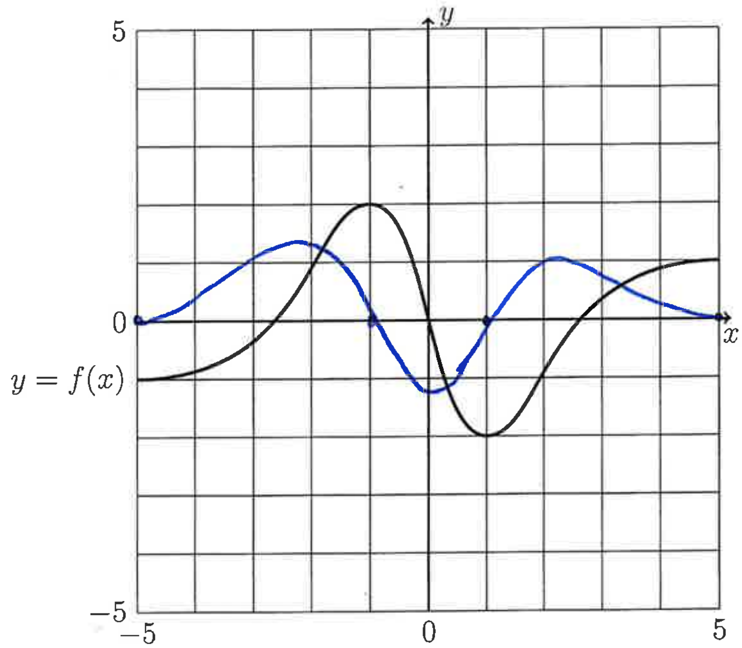
Math 231 Calculus 1 Spring 26 Sample Midterm 2

(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-1, 1)$
(b) Label all regions where $f'(x) > 0$. $(-\infty, -1) \cup (1, \infty)$
(c) Sketch a graph of $f'(x)$ on the figure.
(d) What is $\lim_{x \to \infty} f(x)$? 0
(e) What is $\lim_{x \to -\infty} f'(x)$? 0
(f) Label the approximate locations of all points of inflection. $-2, 0, 2$

(2) Find the derivatives of the following functions
(a) $\sin(2x)e^x + 1/\sqrt{x}$
(b) xe^{-2x^2}
(c) $\dfrac{\cos(2x) - 3}{\sqrt{4x - 1}}$
(d) $x^{\sqrt{x}}$
(e) $\sec(\sqrt{\ln(x)})$
(f) $\sin^{-1}(4/\sqrt[3]{x})$
(g) $\tan^{-1}(3x - 2)$

SMT2 Solutions

Q2 a) $\cos(2x).2e^{x} + \sin(2x)e^{x} + -\frac{1}{2}x^{-3/2}$

b) $e^{-2x^2} + x.e^{-2x^2}.(-4x)$

c) $$\frac{\sqrt{4x-1}.(-2\sin(2x)) + \frac{1}{2}(4x-1)^{-1/2}.(\cos(2x)-3)}{4x-1}.$$

d) $(e^{\ln(x).x^{1/2}})' = e^{\ln(x).x^{1/2}}.\left(\frac{1}{x}.x^{1/2} + \ln(x).\frac{1}{2}x^{-1/2}\right)$

e) $\sec(\sqrt{\ln(x)})\tan(\sqrt{\ln(x)}).\frac{1}{2}(\ln(x))^{-1/2}.\frac{1}{x}$

f) $$\frac{1}{\sqrt{1-\frac{16}{x^{2/3}}}}\cdot 4.-\frac{1}{3}x^{-4/3}.$$

g) $$\frac{1}{1+(3x-1)^2}\cdot 3.$$

Q3 a) $-4\sin(2x).e^{x} + 2\cos(2x)e^{x} + 2\cos(2x)e^{x} + \sin(2x)e^{x} + \frac{3}{4}x^{-5/2}.$

b) $e^{-2x^2}.(-4x) + -8x\,e^{-2x^2} - 4x\,e^{-2x^2}.(-4x).$

Q4 $3y^2 = 4x^2+8$ implicit diffn: $6yy' = 8x$

at $(-1,2)$: $12y' = -8$ $y' = -\frac{2}{3}.$ $y-2 = -\frac{2}{3}(x+1)$

Q5 $$y^2 + x2yy' = \frac{y-y'x}{y^2} + \cos(x-y).(1-y')$$

$$y'\left(2xy + \frac{x}{y^2} + \cos(x-y)\right) = -y^2 + \frac{1}{y} + \cos(x-y)$$

$$y' = \frac{-y^2 + \frac{1}{y} + \cos(x-y)}{2xy + \frac{x}{y^2} + \cos(x-y)}$$

Q6 $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\quad 2 = 4\pi \cdot 16 \cdot \frac{dr}{dt}$ $\quad \frac{dr}{dt} = \frac{1}{32\pi}$.

$A = 4\pi r^2$ $\quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$ $\quad \frac{dA}{dt} = \frac{8\pi \cdot 4}{32\pi} = 1 \text{ cm}^2/\text{sec}$.

Q7 $f(x) = x^{1/3}$. $\quad f(x+h) \simeq f(x) + f'(x)h$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$\sqrt[3]{62} \simeq \sqrt[3]{64} + \frac{1}{3 \cdot \sqrt[3]{64^2}} \cdot (-2)$

$\simeq 4 - \frac{2}{3 \cdot 16} = 4 - \frac{1}{24}$

percentage error: $\frac{\left|\sqrt[3]{62} - \left(4 - \frac{1}{24}\right)\right|}{\sqrt[3]{62}} \times 100 \approx 0.011$

Q8 $f(x) = e^{-2x}(x^2 + 2x + 1)$

$f'(x) = -2e^{-2x}(x^2+2x+1) + e^{-2x}(2x+2) = e^{-2x}(-2x^2 - 2x)$

critical points solve $f'(x) = 0$: $-2x(x+1) = 0$ $\quad x = 0, -1$

		-1		0	
$-2e^{-2x}$	$-$		$-$		$-$
x	$-$		$-$		$+$
$x+1$	$-$		$+$		$+$
$f'(x)$	$-$		$+$		$-$

-1 local min

$+1$ local max

Q9 $f(x) = 2x^2 + 3x - 2$

$f'(x) = 4x + 3$. critical point $x = -\frac{3}{4}$

check $f(-1) = -3$

$f\left(-\frac{3}{4}\right) = 2 \cdot \frac{9}{4} - \frac{9}{4} - 2 = \frac{9}{4}$ $\quad -3\frac{1}{4}$ abs min

$f(1) = 3 \leftarrow$ abs max.

Q10 $f(x) = \frac{e^{-x}}{x^2 - 9}$ $\quad f'(x) = \frac{(x^2-9) \cdot -e^{-x} - 2x \cdot e^{-x}}{(x^2-9)^2}$

a) vertical asymptotes $x = \pm 3$ horizontal asymptotes $\lim_{x \to +\infty} \frac{e^{-x}}{x^2 - 9} \overset{L'H}{=} \lim_{x \to +\infty} \frac{-e^{-x}}{2x} \overset{L'H}{=} \lim_{x \to +\infty} \frac{e^{-x}}{2} = 0$

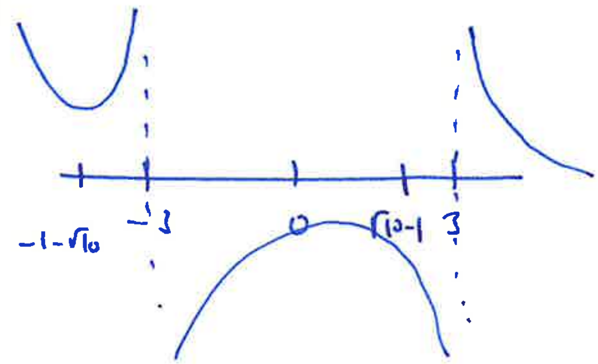
$\lim_{x \to -\infty} \frac{e^{-x}}{x^2-9} = \lim_{x \to \infty} \frac{e^{x}}{x^2 - 9} \overset{L'H}{=} \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = +\infty$ horizontal asymptote $y = 0$.

b) solve $f'(x) = 0$: $-\frac{e^{-x}}{(x^2-9)^2}(x^2 - 9 + 2x) = 0$ $\quad x = \frac{-2 \pm \sqrt{4+36}}{2}$ ~~no solutions.~~ $\quad x = -1 \pm \sqrt{10}$

c)

$\frac{-e^{-x}}{(x^2-9)^2}$	$-$	$-$	$-$	$-$
x^2+2x-9	$+$		$-$	$+$
$f'(x)$	$-$	$-1-\sqrt{10}$ $+$	$-1+\sqrt{10}$	$-$

e)



d) skip – use 1st derivative test

Q11 $f'(x)<0$ max at 1.

Q12 a) $\lim\limits_{x\to-\infty} \frac{\sqrt{2+3x^2}}{4x-1} = \lim\limits_{x\to\infty} \frac{\sqrt{2+3x^2}}{-4x-1} = \lim\limits_{x\to\infty} \frac{\sqrt{\underbrace{2/x^2}_{\to 0}+3}}{-4-\underbrace{1/x}_{\to 0}} = -\frac{\sqrt{3}}{4}$.

l'H:

b) $\lim\limits_{x\to0} \frac{\frac{1}{\sqrt{1-(x/2)^2}}\cdot\frac{1}{2}}{\frac{-1}{\sqrt{1-16x^2}}\cdot 4} = -\frac{4}{3}$.

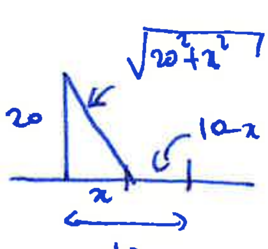
c) $\lim\limits_{x\to0} \frac{\ln(x)}{\csc(x)} \overset{l'H}{=} \lim\limits_{x\to0} \frac{1/x}{-\csc(x)\cot(x)} = \lim\limits_{x\to0} -\frac{\sin(x)\tan(x)}{x}$

$\overset{l'H}{=} \lim\limits_{x\to0} \frac{\cos(x)\tan x + \sin x \sec^2 x}{-1} = 0$

d) $\lim\limits_{x\to0} \frac{(e^{3x}-1)-\sin 3x}{\sin 3x\cdot(e^{3x}-1)} \overset{l'H}{=} \lim\limits_{x\to0} \frac{3e^{3x}-3\cos 3x}{3\cos 3x\,(e^{3x}-1)+\sin 3x\cdot 3e^{3x}}$

$\overset{l'H}{=} \lim\limits_{x\to0} \frac{9e^{3x}+9\sin x}{-9\sin 3x\,(e^{3x}-1)+3\cos 3x\cdot(e^{3x})+3\cos 3x\cdot 3e^{3x}+\sin 3x\, 9e^{3x}} = \frac{9}{6} = \frac{3}{2}$.

Q13



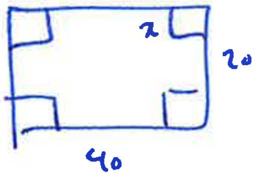
$T = \frac{\sqrt{400+x^2}}{v} + \frac{10-x}{3v}$

$\frac{dT}{dx} = \frac{\frac{1}{2}(20^2+x^2)^{-1/2}\cdot 2x}{v} - \frac{1}{3v}$

critical point soln $\frac{dT}{dx}=0$: $\frac{x}{\sqrt{20^2+x^2}} = 1/3$ $\quad 3x = \sqrt{20^2+x^2}$. $\quad 9x^2 = 20^2+x^2$.

~~$x^2-9x+20^2=0$ $x = 9\pm\sqrt{81}$~~ $\quad 8x^2 = 20^2$ $\quad x^2 = \frac{20^2}{8}$ $\quad x = \frac{20}{2\sqrt{2}}$.

Q14



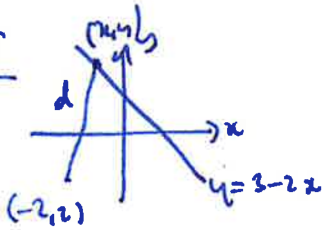
$$V = x.(40-x)(20-x)$$

$$\frac{dV}{dx} = (40-x)(20-x) + x(-1)(20-x) + x(40-x)(-1)$$

$$= 800 - 60x + x^2 \quad = \quad 800 - 120x + 3x^2$$
$$-20x + x^2$$
$$-40x + x^2$$

$$x = \frac{120 \pm \sqrt{120^2 - 4.800.3}}{6} \quad \simeq 8.45$$

Q15



$$d^2 = (x+2)^2 + (y+2)^2 = (x+2)^2 + (3-2x+2)^2 = (x+2)^2 + (5-2x)^2$$

$$= \cancel{x^2+4x}$$

$$\frac{d}{dx}(d^2) = 2(x+2) + 2(5-2x).-2 = \begin{matrix} 2x+4 \\ 4x-20 \end{matrix} = 6x - 16$$

critical pt $x = \frac{16}{6} = \frac{8}{3}$.