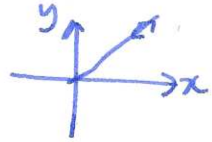
<u>problem</u>: many different parameterizations give the same curve.

e.g. $c(t) = (2t, 2t)$ gives $y = x$, as does $c(t) = (t^3, t^3)$.

<u>warning</u>: $c(t) = (t^2, t^2)$ <u>not</u> all of $y = x$!

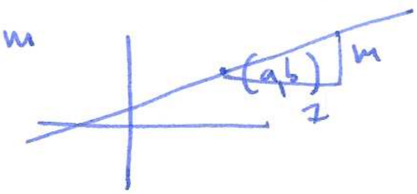



<u>Example</u> describe the parameterized curve $c(t) = (t+1, t^2+1)$ in terms of $y = f(x)$

$$\left.\begin{array}{l} x = t+1 \\ y = t^2+1 \end{array}\right\} \quad t = x-1$$

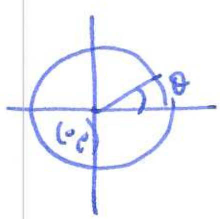
$$y = (x-1)^2 + 1 = x^2 - 2x + 2$$

<u>Example</u> ① find a parametric equation for the line through $(a,b)$ with slope $m$

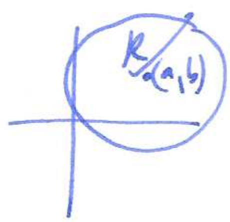



$$\left.\begin{array}{l} t=0: \ (a,b) \\ t=1: \ (a+1, b+m) \end{array}\right\} \ c(t) = (a+t, b+tm)$$

② circle of radius $R$, centered at $(a,b)$

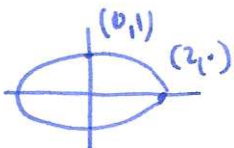



$(R\cos\theta, R\sin\theta)$




$c(\theta) = (a + R\cos\theta, b + R\sin\theta)$

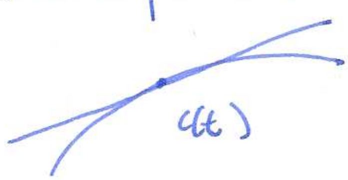
③ ellipse




$c(\theta) = (2\cos\theta, \sin\theta)$

<u>Q</u>: how do we find the slope of the tangent line?

<u>A</u>: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$



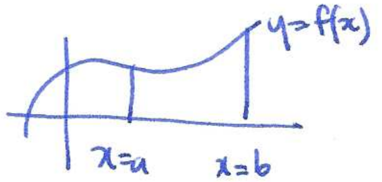

why: $c(t) = (x(t), y(t))$ chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} \Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

<u>Example</u> $c(t) = (\cos t, \sin 2t)$

$$\left.\begin{array}{ll} x(t) = \cos t & y(t) = \sin 2t \\ \dfrac{dx}{dt} = -\sin t & \dfrac{dy}{dt} = 2\cos 2t \end{array}\right\} \ \frac{dy}{dx} = \frac{2\cos 2t}{-\sin t}$$

# §11.2 Arc length and speed

recall



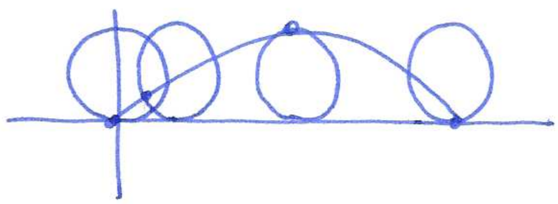

arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

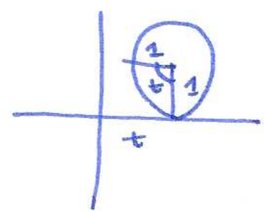

arc length $= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$

$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i$

$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

**Example** cycloid






$x(t) = t-\sin t$ $\qquad \frac{dx}{dt} = 1-\cos t$

$y(t) = 1-\cos(t)$ $\qquad \frac{dy}{dt} = \sin t$

arc length: $\int_0^{2\pi} \sqrt{(1-\cos t)^2+\sin^2 t}\,dt = \int_0^{2\pi}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt$

$= \int_0^{2\pi}\sqrt{2-2\cos t}\,dt$ $\qquad$ recall: $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1-2\sin^2\theta$

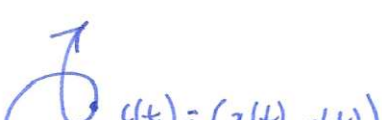
so $2\sin^2\theta = 1-\cos 2\theta$

$2\sin^2\left(\frac{t}{2}\right) = 1-\cos t$

$= \int_0^{2\pi}\sqrt{4\sin^2\left(\frac{t}{2}\right)}\,dt = \int_0^{2\pi} 2\left|\sin\frac{t}{2}\right|dt = 4\int_0^{\pi}\sin\frac{t}{2}\,dt = 4\left[-2\cos\frac{t}{2}\right]_0^{\pi}$

$= 8(0+1) = 8$

speed:

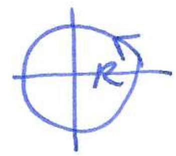



speed $= \frac{d}{dt}(\text{arc length}) = \frac{d}{dt}\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ = length of velocity vector.

**Example** ① $c(t) = (t, t^2)$ $\quad x(t) = t$, $y(t) = t^2$ $\quad \frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$ $\quad$ speed $s'(t) = \sqrt{1+4t^2}$

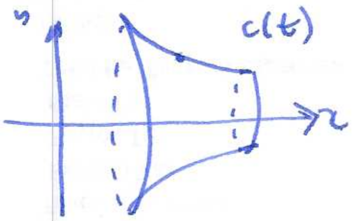
② $c(t) = (R\cos(\omega t), R\sin(\omega t))$




$$c'(t) = (-\omega R\sin(\omega t), \omega R\cos(\omega t))$$

$$\|c'(t)\| = \sqrt{R^2\omega^2\sin^2(\omega t) + \omega^2 R^2\cos^2(\omega t)} = R|\omega|$$

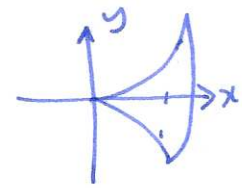
## surface area for parameterized curves




$$\text{surface area} = 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

example $c(t) = (t, t^3)$

$$c'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, 3t^2)$$




$$\text{surface area} = 2\pi \int_0^1 t^3 \sqrt{1^2 + (3t^2)^2}\, dt = 2\pi \int_0^1 t^3\sqrt{1+9t^4}\, dt$$

$u = 1+9t^4$

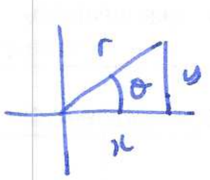
$\frac{du}{dt} = 36t^3$

$$2\pi \int_{\cancel{0}\,1}^{10} t^3\sqrt{u}\,\frac{dt}{du}\, du = 2\pi \int_1^{10} t^3 u^{1/2} \frac{1}{36t^3}\, dt = \frac{\pi}{18}\int_1^{10} u^{1/2}\, du$$

$$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}\left(10^{4/3} - 1\right) \approx 3.56$$

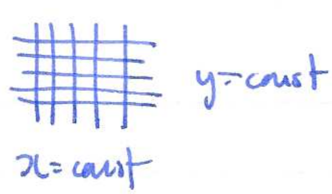
**Example** $c(t) = (t - \tanh t, \operatorname{sech} t)$

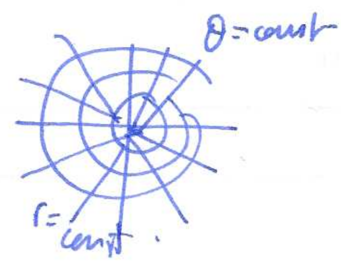
## §10.3 Polar coordinates




$x = r\cos\theta$ $\quad \tan\theta = \frac{y}{x}$

$y = r\sin\theta$ $\quad r^2 = x^2 + y^2$




## Equations in polar coordinates

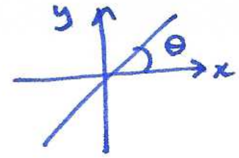
circle of radius $R$ : cartesian $x^2 + y^2 = R^2$

polar $r = R$

straight line through origin: $y = mx$

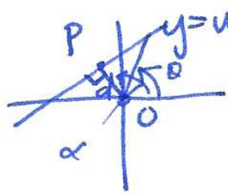
$\theta = \tan^{-1}(m)$

$\theta = \tan^{-1}(m) + \pi$




convention: $(-r, \theta) = (r, \theta + \pi)$, then can just write $\theta = \tan^{-1}(m)$

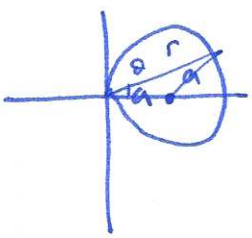
general line:




in polars: let P be closest point to origin (0,0)
and let $d(P, O) = d$

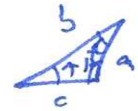
$$\frac{d}{r} = \cos(\theta - \alpha) \qquad r = \frac{d}{\cos(\theta - \alpha)} = d \sec(\theta - \alpha)$$

circle through origin tangent to y-axis



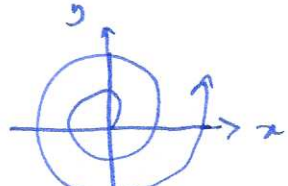

cartesian $(x-a)^2 + y^2 = a^2$

polar $r = 2a\cos\theta$




cosine rule: $a^2 = b^2 + c^2 - 2bc\cos\theta \rightsquigarrow r = 2a\cos\theta$

__Examples__ sketch $r = \theta$



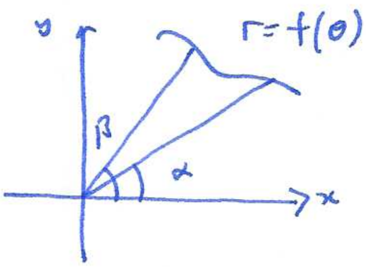

$r = \sin\theta$
$r = \sin 2\theta$ etc.

can always try: $x^2 + y^2 = r^2$
$x = r\cos\theta \quad \leftrightarrow \quad \tan y/x = \theta$
$y = r\sin\theta$

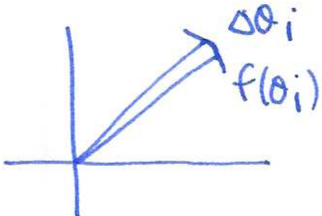
$y = x^2 \rightsquigarrow r\sin\theta = r^2\cos^2\theta$

$$r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$$

## § 11.4 Area and lengths in polars

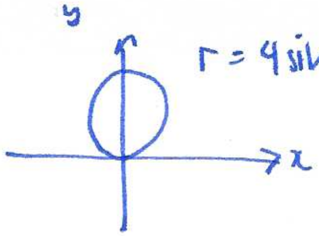



$$\text{area} = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2 \, d\theta$$




$$\text{area } \Delta A \approx \frac{\pi r^2}{2\pi/\Delta\theta_i} = \frac{1}{2} r^2 \Delta\theta_i \qquad \sum \frac{1}{2} r^2 \Delta\theta_i \to \int \frac{1}{2} r^2 \, d\theta$$

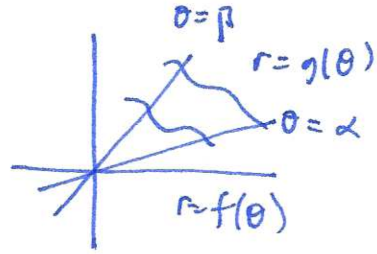
**Example**




$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(4\sin\theta)^2\,d\theta = \int_0^{\pi/2} 8\sin^2\theta\,d\theta$$

$$= 8\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \Big[4\theta - 2\sin 2\theta\Big]_0^{\pi/2} = 4\cdot\frac{\pi}{2} - 0 = 2\pi$$

area between two curves




$$\text{area} = \frac{1}{2}\int_\alpha^\beta (g(\theta))^2 - (f(\theta))^2\,d\theta$$

**arc length** $r = f(\theta)$ is parameterized curve with
$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

so
$$\frac{dx}{d\theta} = f'(\theta)\cos\theta + f(\theta)(-\sin\theta)$$
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

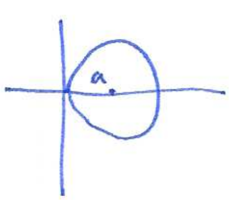
arc length
$$s = \int_\alpha^\beta \sqrt{(f'\cos\theta - f\sin\theta)^2 + (f'\sin\theta + f\cos\theta)^2}\,d\theta$$

$$\left.\begin{array}{l}(f')^2\cos^2\theta - 2ff'\cos\theta\sin\theta + f^2\sin^2\theta \\ (f')^2\sin^2\theta + 2ff'\sin\theta\cos\theta + f^2\cos^2\theta\end{array}\right\} (f')^2 + f^2$$

so arc length in polar is
$$s = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\,d\theta$$

**Example** circle $r = 2a\cos\theta$

$$f(\theta) = 2a\cos\theta$$
$$f'(\theta) = -2a\sin\theta$$




$$\int_0^\pi \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\,d\theta = \int_0^\pi 2a\,d\theta = \Big[2a\theta\Big]_0^\pi = 2\pi a$$