Bad example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1$

Thm Root test (a_n) sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

1. if $L<1$ then $\sum a_n$ converges absolutely
2. if $L>1$ then $\sum a_n$ diverges
3. if $L=1$ no information

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

§10.6 Power series

Defn A power series centered at $x=a$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

note: if this converges, then this gives a function of x.

The series always converges for $x=a$! $F(a) = a_0$

Thm Radius of convergence Let $F(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then

1. $F(x)$ converges only for $x=a$ $(R=0)$

or 2. $F(x)$ converges for all $x \in \mathbb{R}$ $(R=\infty)$

or 3. there is an $R>0$ s.t. $F(x)$ converges for all $|x-a|<R$ and diverges for all $|x-a|>R$. It may or may not converge at $x+R, x-R$.

R is called the radius of convergence.

Example for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2}$

so converges for $\frac{|x|}{2} < 1 \iff |x| < 2$ so $R = 2$.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-5)^{n+1}}{n+1} \cdot \frac{n}{(x-5)^n}\right| = \lim_{n\to\infty} \left|(x-5)\frac{n}{n+1}\right| = |x-5| < 1$

so converges for $|x-5|<1$, $R=1$

Examples $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n!\,x^n$

Thm: Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R>0$, then $f(x)$ is differentiable on $(a-R, a+R)$ and furthermore:

$$\left.\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} a_n n (x-a)^{n-1} \\ \int f(x)\,dx &= c + \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}\right\} \text{ with } \underline{\text{same}} \text{ radius of convergence.}$$

Example ① $1+x+x^2+x^3+\dots = \frac{1}{1-x}$ radius of convergence $R=1$

so $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots$

and $\int \frac{1}{1-x}\,dx = -\ln|1-x| = c + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

② $1-x^2+x^4-x^6\pm\dots = \frac{1}{1+x^2}$

so $\tan^{-1}(x) = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ $\Big\}$ Q: radius of convergence?

③ Fact: $y=e^x$ is the (unique & up to scalar multiple) solution to $\frac{dy}{dx}=y$ $(f'(x)=f(x))$

$\frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

so $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$ radius of convergence $R=\infty$

How to find a power series solution to a differential equation:

try: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Example $y'=y$: $a_1 + 2a_2 x + 3a_3 x^2 + \dots = a_0 + a_1 x + a_2 x^2 + \dots$

$a_1 = a_0$

$2a_2 = a_1 \Rightarrow a_2 = a_0/2$

$3a_3 = a_2 \Rightarrow a_3 = a_0/3!$ so $y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

<u>Example</u> $y'' + y' + y = 0$ with $y(0) = 1$, $y'(0) = 1$

look for a solution $y = a_0 + a_1 x + a_2 x^2 + \cdots$ $y(0) = 1 \Rightarrow a_0 = 1$
$y'(0) = 1 \Rightarrow a_1 = 1$

$y = 1 + x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = 1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = 2a_2 + 6a_3 x + \cdots$

$y'' + y' + y' = (1 + 1 + 2a_2) + x(1 + 2a_2 + 6a_3) + x^2(-\ ) + \cdots = 0$ $a_2 = -1$, $a_3 = -1/6$, $\vdots$

<u>§10.7 Taylor series</u> suppose $f(x)$ has a power series expansion at $x = c$, i.e.

~~suppose~~ $f(x) = a_0 + a_1(x - c) + a_2(x-c)^2 + \cdots$

<u>Q</u>: how do we find the a_i?

<u>A</u>: $f(c) = a_0$ so $a_0 = f(c)$

$f'(x) = a_1 + 2a_2(x - c) + 3a_3(x-c)^2 + \cdots$

$f'(c) = a_1$ so $a_1 = f'(c)$

$f''(x) = 2a_2 + 6a_3(x - c) + \cdots$

$f''(c) = 2a_2$ so $a_2 = \frac{f''(c)}{2}$

<u>Fact</u>: $\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$ <u>Examples</u> $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

<u>Thm</u> If $f(x)$ is equal to a power series centered at $x = c$, with radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$