Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2+9}}$ converge? compare with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+9}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+9}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+9/n^2}} = 1$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}}$ diverges, by limit comparison test.

§10.4 Absolute and conditional convergence

Q: what about $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ ($\star$)

Def$^{\underline{n}}$ A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

($\star$) is absolutely convergent.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ not absolutely convergent.

Th$^{\underline{m}}$ Absolute convergence $\Rightarrow$ convergence.

Proof $0 \leq a_n + |a_n| \leq 2|a_n|$

$$\sum_{n=1}^{\infty} 2|a_n| = \underbrace{2\sum_{n=1}^{\infty} |a_n|}_{\text{converges}} \Rightarrow \sum_{n=1}^{\infty} a_n + |a_n| \text{ converges, by comparison test.}$$

then
$$\underbrace{\sum_{n=1}^{\infty} a_n + |a_n| - |a_n|}_{\text{converges.}} \Leftarrow = \underbrace{\sum_{n=1}^{\infty} a_n + |a_n|}_{\text{converges.}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges.}}$$

So $\sum_{n=1}^{\infty} a_n$ converges $\square$.

Q: what about $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

Def$^{\underline{n}}$ $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge

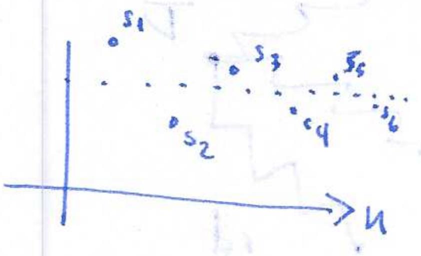
<u>Thm</u> (Alternating series test) Let a_n be a positive, decreasing sequence with $a_n \to 0$. Then $\underbrace{\sum_{n=1}^{\infty} (-1)^n a_n}_{s}$ converges. Furthermore $0 \le s \le a_1$ and $s_{2n} \le s \le s_{2n+1}$ for all n

<u>Proof</u> • even partial sums: $s_{2n} = \underbrace{a_1 - a_2}_{\geq 0} + \underbrace{a_3 - a_4}_{\geq 0} + \cdots + \underbrace{a_{2n-1} - a_{2n}}_{\geq 0}$

positive increasing sequence.

• odd partial sums: $s_{2n+1} = a_1 - \underset{\geq 0}{(a_2 - a_3)} - \underset{\geq 0}{(a_4 - a_5)} - \cdots - \underset{\geq 0}{(a_{2n} - a_{2n+1})}$

decreasing sequence.



furthermore $s_{2n} = a_1 - (a_2 - a_3) - \cdots - a_{2n}$

so $s_{2n} \le a_1$ for all n

so s_{2n} is an increasing sequence, bounded above,

so $\lim_{n\to\infty} s_{2n}$ exists.

similarly $\lim_{n\to\infty} s_{2n+1}$ exists.

finally: $\lim_{n\to\infty} s_{2n} - s_{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} -a_{2n+1} = 0$

so $\lim_{n\to\infty} s_n$ exist. $\square$.

<u>Example</u> show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges (alternating harmonic series)

use alternating series test $a_n = \frac{1}{n}$
- positive
- decreasing
- $\lim_{n\to\infty} a_n = 0$

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. $\square$.

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ conditionally convergent.

§10.5 Ratio and root tests

Fact $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$ Q: how do we show this converges?

(e.g. use comparison test $n! = 1.2. \dots (n-2)(n-1)n > (n-1)^2$ so $\frac{1}{n!} < \frac{1}{(n-1)^2}$)

Thm **Ratio test** (a_n) sequence, and suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ exists.

Then ① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

③ if $\rho = 1$ no information

Proof if $\rho < 1$, then there is a number $\rho < r < 1$, and a number N

s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$, so $|a_{N+1}| \leq r|a_N|$

$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$ etc.

so $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^{n-N} \leq \frac{|a_N|}{1-r}$ so converges by comparison test with geometric series.

If $\rho > 1$, then there is $\rho > r > 1$ and N s.t. $\left|\frac{a_{N+1}}{a_N}\right| > r > 1$

for all $n \geq N$ and so $a_n \not\to 0 \Rightarrow$ diverges. $\square$.

Example ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

② show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{n^3/3^n} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$