$$s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}.$$

$$s_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}$$

$$\left.\begin{aligned} S_N &= \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^N} \\ \frac{1}{2}S_N &= \quad\;\; \frac{1}{4}+\cdots \quad \frac{1}{2^N}+\frac{1}{2^{N+1}} \end{aligned}\right\} \quad S_N - \frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$\frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}} \quad \Rightarrow \quad S_N = 1-\frac{1}{2^N} \quad \text{so} \quad \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1-\frac{1}{2^N} = 1$$

<u>General case</u>

$$c + cr + cr^2 + cr^3 + \ldots = \sum_{n=0}^{\infty} cr^n$$

$$S_N = c + cr + cr^2 + \cdots + cr^N$$

$$rS_N = \quad cr + cr^2 + \cdots + cr^{N} + cr^{N+1}$$

$$S_N - rS_N = c - cr^{N+1}$$

$$S_N(1-r) = c(1-r^{N+1})$$

$$S_N = \frac{c}{1-r}\,1-r^{N+1} \qquad \lim_{N\to\infty} c\,\frac{1-r^{N+1}}{1-r} = \frac{c}{1-r} \quad \text{if } |r|<1$$

otherwise <u>does not converge</u>.

<u>Special cases</u> (Telescoping)

<u>Example</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots$

use: $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$

$$s_1 = 1-\frac{1}{2}$$

$$s_2 = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3} = 1-\frac{1}{3}.$$

$$s_3 = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4} = 1-\frac{1}{4}$$

so $\displaystyle S_N = 1-\frac{1}{N+1} \qquad \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1-\frac{1}{N} = 1$

<u>Example</u> (harmonic series)

diverges!

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{17}+\cdots+\frac{1}{32}}_{\geq\frac{1}{2}}+\cdots$$

Q: when does a series converge? A: dunno

Tools

- Thm (Divergence test) if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning: $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Recall $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

Rules for series / infinite sums

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series

then
$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty} c a_n = c\sum_{n=1}^{\infty} a_n$$

observation: same series can be written in many ways.
$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{(\sqrt{2})^{2n}}$$

§10.3 Positive series

positive series: $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots$ all $a_n \geq 0$

note the sequence of partial sums S_N is increasing: $S_{N+1} = S_N + \underbrace{a_{n+1}}_{\geq 0}$

recall: an increasing sequence with an upper bound converges.

Thm Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs:

① the partial sums are bounded above, and S converges.

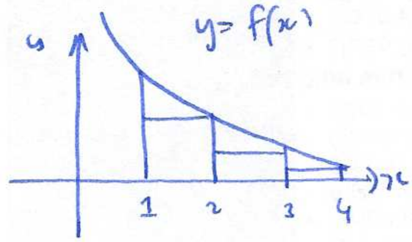
② the partial sums are not bounded above, and S diverges.

Thm (Integral test) Let $a_n = f(n)$ where $f(x)$ is $\left.\begin{array}{l}\text{positive}\\ \text{decreasing}\\ \text{continuous}\end{array}\right\}$ for $x \geq 1$.

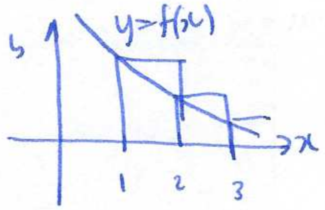
then ① if $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges

② if $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Proof



$$S_N - a_1 = a_2 + a_3 + a_4 + \cdots \leq \int_1^N f(x)\,dx$$



$$S_N = a_1 + a_2 + \cdots \geq \int_1^N f(x)\,dx \qquad \square.$$

Example $s = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\int_1^\infty x^{-1/2}\,dx = \lim_{R\to\infty} \int_1^R x^{-1/2}\,dx = \lim_{R\to\infty} \left[2x^{1/2}\right]_1^R = \lim_{R\to\infty} 2\sqrt{R} - 2 \to \infty \text{ as } N \to \infty$$

so s diverges.

Thm (Convergence of p-series) $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

Proof (integral test) $\sum_{n=1}^\infty a_n \quad a_n = f(n), \; f(x) = \frac{1}{x^p} = x^{-p}$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \leq 1$ $\square$.

Example $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^\infty \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$)

$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum_{n=1}^\infty \frac{1}{n^3}$ converges (to ?).

Thm <u>Comparison test</u> suppose $0 \le a_n \le b_n$ for all $n \ge M$

then ① if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

② if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

<u>Example</u> $\sum_{n=1}^{\infty} 2^{-n^2} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \dots$

note: $\frac{1}{2^{n^2}} < \frac{1}{2^n}$, geometric series converges.

Thm <u>Limit comparison test</u> a_n, b_n <u>positive</u> sequences.

suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$ exists

then · if $L > 0$ $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges

· if $L = 0$ $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

<u>Proof</u> (sketch) case $L > 0$, then $\frac{a_n}{b_n} \to L$, so $0 < \frac{a_n}{b_n} < R$ for some $L < R$.

$0 < a_n < R b_n$

comparison test: $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

similarly $\frac{b_n}{a_n} \to \frac{1}{L}$, so $0 < \frac{b_n}{a_n} < R'$ for some $\frac{1}{L} < R'$

$0 < b_n < R' a_n$, so comparison test: $\sum a_n$ converges $\Rightarrow$ $\sum b_n$ converges.

(if $L = 0$ only get one direction) □

<u>Example</u> show $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges. (for large n, $\sim \frac{1}{n^2}$)

compare with $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2}{n^4 - n - 1}}{1/n^2} = \frac{n^4}{n^4 - n - 1} = \frac{1}{1 - 1/n^3 - 1/n^4} = 1$