§10.1 Sequences

Def^n A sequence is a list of numbers indexed by $\mathbb{N}$ = positive integers.

Example $1, 2, 3, 4, 5, 6, \dots$ Notation $a_1, a_2, a_3, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ or $(a_n)_{n \in \mathbb{N}}$

$1, 27, \pi, \sqrt{2}, \dots$ where a_n is the n-th number in the sequence

$1, 1, 1, 1, \dots$

Q: what is not a sequence? , a single number, a set of numbers, a function …

sometimes (but not always) we can give the sequence by a formula

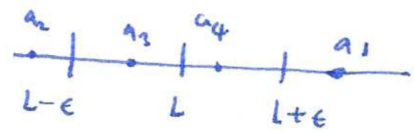
Examples $(a_n)_{n \in \mathbb{N}}$ $a_n = n$ $(n)_{n \in \mathbb{N}} = (1, 2, 3, 4, \dots)$

$(a_n)_{n \in \mathbb{N}}$ $a_n = \frac{1}{1+n^2}$ $\left(\frac{1}{1+n^2}\right)_{n \in \mathbb{N}} = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots\right)$

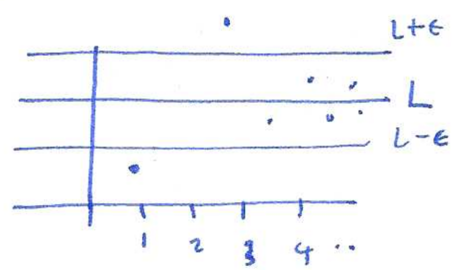
Example (recursive def^n)

$a_{n+2} = a_{n+1} + a_n$ $a_1 = 1, a_2 = 1$ gives $1, 1, 2, 3, 5, 8, 13, \dots$ (Fibonacci sequence)

Def^n A sequence (a_n) converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| \leq \epsilon$ for all $n \geq N$



Notation $\lim_{a_n \to L} a_n = L$ or $a_n \to L$



Example $a_n = \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$

Proof given $\epsilon > 0$; choose $N > \frac{1}{\epsilon}$, then if $n > N$, then $\frac{1}{n} < \frac{1}{N} < \epsilon$, as required $\square$.

special case: sequence defined by a formula / function $a_n = f(n)$

Th^m If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} f(n) = L$

Q is the converse true? (No)

Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim a_n = 1$.

Example (geometric sequence) $a_n = r^n$

e.g.: $2, 4, 8, 16, \ldots \qquad a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots \qquad a_n = \frac{1}{3^n}$

$1, 1, 1, \ldots \qquad a_n = 1$

$-1, +1, -1, +1, \ldots \qquad a_n = (-1)^n$

Fact $\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \\ \text{DNE} & r \le -1 \end{cases}$

Rules for limits of sequences: same as rules for limits of functions.

suppose $a_n \to L$ and $b_n \to M$, then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (a_n) + \lim_{n\to\infty} (b_n) = L + M$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\left(\lim_{n\to\infty} a_n\right)}{\lim_{n\to\infty} b_n} = \frac{L}{M}$ as long as $M \neq 0$!
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ (c constant, does not depend on n)

Squeeze Thm If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

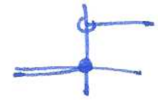
Example $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any $R \in \mathbb{R}$

Proof there is an integer M such that $M \le R \le M+1$

$$0 \le \frac{R^n}{n!} = \underbrace{\frac{R}{1} \cdot \frac{R}{2} \cdot \ldots \cdot \frac{R}{M}}_{\text{call this } A} \underbrace{\frac{R}{M+1} \cdots \frac{R}{n-1}}_{\le 1} \cdot \frac{R}{n} \le A \frac{R}{n}$$

so $0 \le \frac{R^n}{n!} \le A \cdot \frac{R}{n}$, $\lim_{n\to\infty} 0 = 0$ $\lim_{n\to\infty} \frac{AR}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{R^n}{n!} = 0$ $\square$.

Thm If $f(x)$ is cts and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$

Important f continuous (at L). Bad example $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$ 

then $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) = 1$ for all n and $f(0) = 0 \neq \lim_{n\to\infty} f(\frac{1}{n}) = 1$.

Example find $\lim_{n\to\infty} e^{n/n+1}$

start with $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$, then $\lim_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$.

Defn A sequence (a_n) is

- bounded above if $a_n \le M$ for all n.
- bounded below if $L \le a_n$ for all n.
- bounded if $L \le a_n \le M$ for all n.

Thm Convergent subsequences are bounded

Warning: bounded subsequences need not converge.

Example: $0, 1, 0, 1, \ldots$ $\quad a_n = \frac{1-(-1)^n}{2}$

Thm Bounded monotonic sequences converge.

- if (a_n) is increasing and $a_n \le M$ then $a_n \to l \le M$
- if (a_n) is decreasing and $L \le a_n$ then $a_n \to l \ge L$

Example $a_n = \frac{1}{n}$. show decreasing want $a_n \ge a_{n+1}$

$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$ ✓ lower bound $L = -100$ $\lim_{n\to\infty} \frac{1}{n} = l \ge -100$.

Example show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below:

note: $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$ as $\sqrt{x}$ monotonically increasing.

$\Rightarrow a_n \ge 0$ so we can choose lower bound $L = 0$.

decreasing: consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{1/2}$

$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$. claim: $f'(x) < 0$.

alternatively $\frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ ← decreasing.

§10.2 Series

Def$^{\underline{n}}$ A series is an infinite sum $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $\quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$1 + 1 + 1 + \cdots$ $\quad 1 - 1 + 1 - 1 + 1 - 1 \cdots$

Def$^{\underline{n}}$ The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

Def$^{\underline{n}}$ The sum of the infinite series is defined to be the limit of the sequence of partial sums, if this limit exists.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

If $\lim_{N\to\infty} S_n = s$ we say $\sum_{n=1}^{\infty} a_n$ converges to s and write $\sum_{n=1}^{\infty} a_n = s$

Example ① $1 + 1 + 1 + \cdots$ $\quad S_N = 1 + 1 + \cdots + 1 = N$ $\quad \lim_{N\to\infty} N = \infty$

so $\sum_{n=1}^{\infty} 1$ does not converge.

② $1 - 1 + 1 - 1 + \cdots$ $\quad s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0, \ldots$

$(S_N) = 1, 0, 1, 0, 1, \ldots$ does not converge.

warning: can't re-arrange non-converging sums: $(1-1)+(1-1)+(1-1)+\cdots = 0$

$0 + 0 + 0 + \cdots$

$1 + (-1+1) + (-1+1) + (-1+1) + \cdots$

$1 + 0 + 0 + 0 + \cdots = 1.$

Geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$s_1 = \frac{1}{2}$